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On some sets of fuzzy-valued sequences with the level sets

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Abstract. In this paper, we introduce the sets of bounded, convergent and null series and the set of sequences of bounded variation of fuzzy numbers with the level sets. We investigate the relationships between these sets and their classical forms and give some properties including definitions, lemmas and various kind of fuzzy metric spaces. Furthermore, we study some of their properties like completeness, duality and present some illustrative examples related to these sets. Finally, we obtain the alpha-, beta- and gamma-duals of the sets of sequences of fuzzy numbers with respect to the level sets.

Key words. Sequence space, metric space, fuzzy metric space, complete metric space.

1 Introduction

By $\omega(F)$ and E^1, we denote the set of all sequences of fuzzy numbers and the set of all fuzzy numbers on $\mathbb{R}$, respectively. We define the classical sets $bs(F)$, $cs(F)$, $cs_0(F)$ and $bv(F)$ consisting of the sets of all bounded, convergent, null series and the set of bounded variation sequences of fuzzy numbers, respectively, that is

$$bs(F) := \left\{ u = (u_k) \in \omega(F) : \sup_{n\in\mathbb{N}} D\left(\sum_{k=0}^{n} u_k, \overline{0}\right) < \infty \right\},$$

$$cs(F) := \left\{ u = (u_k) \in \omega(F) : \exists l \in E^1 \ni \lim_{n\to\infty} D\left(\sum_{k=0}^{n} u_k, \bar{l}\right) = 0 \right\},$$

$$cs_0(F) := \left\{ u = (u_k) \in \omega(F) : \lim_{n\to\infty} D\left(\sum_{k=0}^{n} u_k, \overline{0}\right) = 0 \right\},$$

$$bv(F) := \left\{ u = (u_k) \in \omega(F) : \sum_{k=0}^{\infty} D\left[(\Delta u)_k, \overline{0}\right] < \infty \right\},$$

where $(\Delta u)_k = u_k - u_{k+1}$ for all $k \in \mathbb{N}$. We can show that $bs(F)$, $cs(F)$ and $cs_0(F)$ are complete metric spaces with the metric D_∞ on E^1 defined by means of the Hausdorff metric d as

$$D_\infty(u,v) := \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} D\left(u_k, v_k\right) = \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \sup_{\lambda\in[0,1]} d\big([u_k]_\lambda, [v_k]_\lambda\big),$$

where $u = (u_k)$ and $v = (v_k)$ are the elements of the sets $bs(F)$, $cs(F)$ or $cs_0(F)$. The space $bv(F)$ of sequences of bounded variation is complete metric space with the metric D_Δ defined by

$$D_\Delta(u,v) := \sum_{k=0}^{\infty} D\left[(\Delta u)_k, (\Delta v)_k\right] = \sum_{k=0}^{\infty} \sup_{\lambda\in[0,1]} \left\{ d\left([(\Delta u)_k]_\lambda, [(\Delta v)_k]_\lambda\right)\right\},\ (\Delta u)_k = u_k - u_{k+1}$$

where $u = (u_k)$, $v = (v_k)$ are the elements of the set $bv(F)$.

Many authors have extensively developed the theory of the different cases of sequence sets with fuzzy metric. Mursaleen and Başarır [6] have recently introduced some new sets of sequences of fuzzy numbers generated by a non-negative regular matrix A some of which reduced to the Maddox's spaces $\ell_\infty(p;F)$, $c(p;F)$, $c_0(p;F)$ and $\ell(p;F)$ of sequences of fuzzy numbers for the special cases of that matrix A. Altın, Et and Çolak [23] have recently defined the concepts of lacunary statistical convergence and lacunary strongly convergence of generalized difference sequences of fuzzy numbers. Quite recently; Talo and Başar [12] have extended the main results of Başar and Altay [2] to fuzzy numbers and defined the alpha-, beta- and gamma-duals of a set of sequences of fuzzy numbers, and gave the duals of the classical sets of sequences of fuzzy numbers together with the characterization of the classes of infinite matrices of fuzzy numbers transforming one of the classical set into another one. Also, Kadak and Başar [16–19] have recently studied the power series of fuzzy numbers and examined the alternating and binomial series of fuzzy numbers. Furthermore, Kadak determine some sets of sequences and fuzzy valued functions via fuzzy metric with some inclusion relations, in [20, 21]. Finally, Talo and Başar [11] have introduced the sets $\ell_\infty(F;f)$, $c(F;f)$, $c_0(F;f)$ and $\ell(F;f)$ of sequences of fuzzy numbers defined by a modulus function and the spaces $\ell_\infty(F)$, $c(F)$, $c_0(F)$ and $\ell_p(F)$ of sequences of fuzzy numbers consisting of the bounded, convergent, null and absolutely p-summable sequences of fuzzy numbers with the level sets.

The main purpose of the present paper is to study the corresponding sets $bs(F)$, $cs(F)$, $cs_0(F)$ and $bv(F)$ of sequences of fuzzy numbers. We essentially proceed with some classes between the classical sets of sequences of fuzzy numbers.

The rest of this paper is organized as follows:

In Section 2, some required definitions and consequences related with the fuzzy numbers, sequences and series of fuzzy numbers are given. Section 3 is devoted to the completeness of the sets of sequences $bs(F)$, $cs(F)$, $cs_0(F)$ and $bv(F)$ of fuzzy numbers and some related examples. In the final section of the paper, the alpha-, beta- and gamma-duals of the sets of $bs(F)$, $cs(F)$, $cs_0(F)$ and $bv(F)$ of fuzzy numbers are determined and given some properties including definitions, lemmas and theorems. At the end of Section 4, an example on alpha-, beta- and gamma-duals of $cs(F)$ is given.

2 Preliminaries, background, and notation

A *fuzzy number* is a fuzzy set on the real axis, i.e., a mapping $u : \mathbb{R} \to [0,1]$ which satisfies the following four conditions:

(i) u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$.

(ii) u is fuzzy convex, i.e., $u[\lambda x + (1-\lambda)y] \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}$ and for all $\lambda \in [0,1]$.

(iii) u is upper semi-continuous.

(iv) The set $[u]_0 = \overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is compact, (cf. Zadeh [5]), where $\overline{\{x \in \mathbb{R} : u(x) > 0\}}$ denotes the closure of the set $\{x \in \mathbb{R} : u(x) > 0\}$ in the usual topology of $\mathbb{R}$.

We denote the set of all fuzzy numbers on $\mathbb{R}$ by E^1 and called it as *the space of fuzzy numbers*. λ-*level set* $[u]_\lambda$ of $u \in E^1$ is defined by

$$[u]_\lambda := \begin{cases} \{t \in \mathbb{R} : u(t) \geq \lambda\} & , \quad 0 < \lambda \leq 1, \\ \overline{\{t \in \mathbb{R} : u(t) > \lambda\}} & , \quad \lambda = 0. \end{cases}$$

The set $[u]_\lambda$ is closed, bounded and non-empty interval for each $\lambda \in [0,1]$ which is defined by $[u]_\lambda := [u^-(\lambda), u^+(\lambda)]$. $\mathbb{R}$ can be embedded in E^1, since each $r \in \mathbb{R}$ can be regarded as a fuzzy number $\overline{r}$ defined by

$$\overline{r}(x) := \begin{cases} 1 & , \quad x = r, \\ 0 & , \quad x \neq r. \end{cases}$$

Theorem 2.1 (Representation Theorem) ([14]) *Let* $[u]_\lambda = [u^-(\lambda), u^+(\lambda)]$ *for* $u \in E^1$ *and for each* $\lambda \in [0,1]$*. Then the following statements hold:*

(i) u^- *is a bounded and non-decreasing left continuous function on* $]0,1]$.

(ii) u^+ *is a bounded and non-increasing left continuous function on* $]0,1]$.

(iii) *The functions* u^- *and* u^+ *are right continuous at the point* $\lambda = 0$.

(iv) $u^-(1) \leq u^+(1)$.

Conversely, if the pair of functions u^- *and* u^+ *satisfies the conditions (i)-(iv), then there exists a unique* $u \in E^1$ *such that* $[u]_\lambda := [u^-(\lambda), u^+(\lambda)]$ *for each* $\lambda \in [0,1]$*. The fuzzy number* u *corresponding to the pair of functions* u^- *and* u^+ *is defined by* $u : \mathbb{R} \to [0,1]$, $u(x) := \sup\{\lambda : u^-(\lambda) \leq x \leq u^+(\lambda)\}$.

Now we give the definitions of the well-known two types of fuzzy numbers with the λ-level set.

Definition 2.1 (Triangular Fuzzy Number) ([4, Definition, p. 137]) *We can define the triangular fuzzy number* u *as* $u = (u_1, u_2, u_3)$ *whose membership function* $\mu_{(u)}$ *is interpreted as follows;*

$$\mu_{(u)}(x) = \begin{cases} \frac{x-u_1}{u_2-u_1} & , \quad u_1 \leq x \leq u_2, \\ \frac{u_3-x}{u_3-u_2} & , \quad u_2 \leq x \leq u_3, \\ 0 & , \quad x < u_1, \quad x > u_3. \end{cases}$$

Then, the result $[u]_\lambda := [u^-(\lambda), u^+(\lambda)] = [(u_2 - u_1)\lambda + u_1, -(u_3 - u_2)\lambda + u_3]$ *holds for each* $\lambda \in [0,1]$.

Definition 2.2 (Trapezoidal Fuzzy Number) ([4, Definition, p. 145]) *We can define the trapezoidal fuzzy number* u *as* $u = (u_1, u_2, u_3, u_4)$ *whose membership function* $\mu_{(u)}$ *is interpreted as follows:*

$$\mu_{(u)}(x) = \begin{cases} \frac{x-u_1}{u_2-u_1} & , \quad u_1 \leq x \leq u_2, \\ 1 & , \quad u_2 \leq x \leq u_3, \\ \frac{u_4-x}{u_4-u_3} & , \quad u_3 \leq x \leq u_4, \\ 0 & , \quad x < u_1, \quad x > u_4. \end{cases}$$

Then, the result $[u]_\lambda := [u^-(\lambda), u^+(\lambda)] = [(u_2 - u_1)\lambda + u_1, -(u_4 - u_3)\lambda + u_4]$ *holds for each* $\lambda \in [0,1]$.

Let $u, v, w \in E^1$ and $\alpha \in \mathbb{R}$. Then the operations addition, scalar multiplication and product defined on E^1 by

$$
\begin{aligned}
u + v = w \quad &\Leftrightarrow \quad [w]_\lambda = [u]_\lambda + [v]_\lambda \text{ for all } \lambda \in [0,1] \\
&\Leftrightarrow \quad w^-(\lambda) = u^-(\lambda) + v^-(\lambda) \ \text{ and } w^+(\lambda) = u^+(\lambda) + v^+(\lambda) \ \text{ for all } \ \lambda \in [0,1], \\
&\qquad [\alpha u]_\lambda = \alpha [u]_\lambda \ \text{ for all } \lambda \in [0,1], \\
&\qquad uv = w \Leftrightarrow [w]_\lambda = [u]_\lambda [v]_\lambda \ \text{ for all } \lambda \in [0,1],
\end{aligned}
$$

where it is immediate that

$$
\begin{aligned}
w^-(\lambda) &= \min\{u^-(\lambda)v^-(\lambda), u^-(\lambda)v^+(\lambda), u^+(\lambda)v^-(\lambda), u^+(\lambda)v^+(\lambda)\}, \\
w^+(\lambda) &= \max\{u^-(\lambda)v^-(\lambda), u^-(\lambda)v^+(\lambda), u^+(\lambda)v^-(\lambda), u^+(\lambda)v^+(\lambda)\}
\end{aligned}
$$

for all $\lambda \in [0,1]$. Let W be the set of all closed bounded intervals A of real numbers with endpoints $\underline{A}$ and $\overline{A}$, i.e., $A := [\underline{A}, \overline{A}]$. Define the relation d on W by

$$d(A, B) := \max\{|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|\}.$$

Then it can easily be observed that d is a metric on W (cf. Diamond and Kloeden [13]) and (W, d) is a complete metric space, (cf. Nanda [15]). Now, we can define the metric D on E^1 by means of the Hausdorff metric d as

$$D(u, v) := \sup_{\lambda \in [0,1]} d([u]_\lambda, [v]_\lambda) := \sup_{\lambda \in [0,1]} \max\{|u^-(\lambda) - v^-(\lambda)|, |u^+(\lambda) - v^+(\lambda)|\}.$$

Definition 2.3 ([10, Definition 2.1]) *$u \in E^1$ is said to be a non-negative fuzzy number if and only if $u(x) = 0$ for all $x < 0$. It is immediate that $u \succeq \overline{0}$ if u is a non-negative fuzzy number.*

One can see that

$$D(u, \overline{0}) = \sup_{\lambda \in [0,1]} \max\{|u^-(\lambda)|, |u^+(\lambda)|\} = \max\{|u^-(0)|, |u^+(0)|\}.$$

Proposition 2.2 ([1]) *Let $u, v, w, z \in E^1$ and $\alpha \in \mathbb{R}$. Then, the following statements hold:*

(i) *(E^1, D) is a complete metric space, (cf. Puri and Ralescu [9]).*

(ii) *$D(\alpha u, \alpha v) = |\alpha| D(u, v)$.*

(iii) *$D(u + v, w + v) = D(u, w)$.*

(iv) *$D(u + v, w + z) \leq D(u, w) + D(v, z)$.*

(v) $|D(u, \overline{0}) - D(v, \overline{0})| \leq D(u, v) \leq D(u, \overline{0}) + D(v, \overline{0})$.

Definition 2.4 ([10, Definition 2.7]) *A sequence $u = (u_k)$ of fuzzy numbers is a function u from the set $\mathbb{N}$ into the set E^1. The fuzzy number u_k denotes the value of the function at $k \in \mathbb{N}$ and is called as the general term of the sequence.*

Definition 2.5 ([10, Definition 2.9]) *A sequence $(u_n) \in \omega(F)$ is called convergent with limit $u \in E^1$, if and only if for every $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that*

$$D(u_n, u) < \varepsilon \quad \textit{for all} \quad n \geq n_0.$$

Obviously the sequence $(u_n) \in \omega(F)$ converges to a fuzzy number u if and only if $\{u_n^-(\lambda)\}$ and $\{u_n^+(\lambda)\}$ converge uniformly to $u^-(\lambda)$ and $u^+(\lambda)$ on $[0, 1]$, respectively.

Definition 2.6 ([10, Definition 2.11]) *A sequence $(u_n) \in \omega(F)$ is called bounded if and only if the set of fuzzy numbers consisting of the terms of the sequence (u_n) is a bounded set. That is to say that a sequence $(u_n) \in \omega(F)$ is said to be bounded if and only if there exist two fuzzy numbers m and M such that $m \preceq u_n \preceq M$ for all $n \in \mathbb{N}$. This means that $m^-(\lambda) \leq u_n^-(\lambda) \leq M^-(\lambda)$ and $m^+(\lambda) \leq u_n^+(\lambda) \leq M^+(\lambda)$ for all $\lambda \in [0, 1]$.*

The boundedness of the sequence $(u_n) \in \omega(F)$ is equivalent to the fact that

$$\sup_{n \in \mathbb{N}} D(u_n, \overline{0}) = \sup_{n \in \mathbb{N}} \sup_{\lambda \in [0,1]} \max\{|u_n^-(\lambda)|, |u_n^+(\lambda)|\} < \infty.$$

If the sequence $(u_k) \in \omega(F)$ is bounded then the sequences of functions $\{u_k^-(\lambda)\}$ and $\{u_k^+(\lambda)\}$ are uniformly bounded in $[0, 1]$.

Theorem 2.3 ([7, Theorem 4.1]) *Let $(u_k), (v_k) \in \omega(F)$ with $u_k \to a$, $v_k \to b$, as $k \to \infty$. Then, the following statements hold:*

(i) $u_k + v_k \to a + b$, *as* $k \to \infty$.

(ii) $u_k - v_k \to a - b$, *as* $k \to \infty$.

(iii) $u_k v_k \to ab$, *as* $k \to \infty$.

(iv) $u_k / v_k \to a/b$, *as* $k \to \infty$*; where* $0 \notin [v_k]_0$ *for all* $k \in \mathbb{N}$ *and* $0 \notin [b]_0$.

Definition 2.7 ([22]) *Let $(u_k) \in \omega(F)$. Then the expression $\sum_{k=0}^{\infty} u_k$ is called a series of fuzzy numbers. Define the sequence (s_n) via nth partial sum of the series by $s_n = u_0 + u_1 + u_2 + \cdots + u_n$*

for all $n \in \mathbb{N}$. If the sequence (s_n) converges to a fuzzy number u then we say that the series $\sum_{k=0}^{\infty} u_k$ of fuzzy numbers converges to u and write $\sum_{k=0}^{\infty} u_k = u$ which implies that

$$\lim_{n\to\infty} \sum_{k=0}^{n} u_k^-(\lambda) = u^-(\lambda) \quad and \quad \lim_{n\to\infty} \sum_{k=0}^{n} u_k^+(\lambda) = u^+(\lambda),$$

uniformly in $\lambda \in [0,1]$. Conversely, if the fuzzy numbers

$$u_k = \left\{(u_k^-(\lambda), u_k^+(\lambda)) : \lambda \in [0,1]\right\}, \quad \sum_{k=0}^{\infty} u_k^-(\lambda) = u^-(\lambda) \quad and \quad \sum_{k=0}^{\infty} u_k^+(\lambda) = u^+(\lambda)$$

converge uniformly in λ, then $u = \{(u^-(\lambda), u^+(\lambda)) : \lambda \in [0,1]\}$ defines a fuzzy number such that $u = \sum_{k=0}^{\infty} u_k$.

We say otherwise the series of fuzzy numbers diverges. As this, if the sequence (s_n) is bounded then we say that the series $\sum_{k=0}^{\infty} u_k$ of fuzzy numbers is bounded.

Definition 2.8 ([10, Definition 2.14]) *Let $\{f_k(\lambda)\}$ be a sequence of functions defined on $[a,b]$ and $\lambda_0 \in]a,b]$. Then, $\{f_k(\lambda)\}$ is said to be eventually equi-left-continuous at λ_0 if for any $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ and $\delta > 0$ such that $|f_k(\lambda) - f_k(\lambda_0)| < \varepsilon$ whenever $\lambda \in]\lambda_0 - \delta, \lambda_0]$ and $k \geq n_0$. Similarly, eventually equi-right-continuity at $\lambda_0 \in [a,b[$ of $\{f_k(\lambda)\}$ can be defined.*

Theorem 2.4 ([10, Theorem 2.15]) *Let (u_k) be a sequence of fuzzy numbers such that $\lim_{k\to\infty} u_k^-(\lambda) = u^-(\lambda)$ and $\lim_{k\to\infty} u_k^+(\lambda) = u^+(\lambda)$ for each $\lambda \in [0,1]$. Then the pair of functions u^- and u^+ determine a fuzzy number if and only if the sequences of functions $\{u_k^-(\lambda)\}$ and $\{u_k^+(\lambda)\}$ are eventually equi-left-continuous at each $\lambda \in]0,1]$ and eventually equi-right-continuous at $\lambda = 0$.*

Thus, it is deduced that the series $\sum_{k=0}^{\infty} u_k^-(\lambda) = u^-(\lambda)$ and $\sum_{k=0}^{\infty} u_k^+(\lambda) = u^+(\lambda)$ define a fuzzy number if the sequences

$$\{s_n^-(\lambda)\} = \left\{\sum_{k=0}^{n} u_k^-(\lambda)\right\} \quad \text{and} \quad \{s_n^+(\lambda)\} = \left\{\sum_{k=0}^{n} u_k^+(\lambda)\right\}$$

satisfy the conditions of Theorem 2.4.

Definition 2.9 ([8]) *A sequence $\{u_n(x)\}$ of fuzzy valued functions converges uniformly to $u(x)$ on a set I if for each $\varepsilon > 0$ there exists a number n_0 such that $D(u_n(x), u(x)) < \varepsilon$ for all $x \in I$ and $n > n_0$.*

It is clear that if (u_n) is uniformly convergent to u, then the sequence is pointwise convergent to u on I. But pointwise convergence of (u_n) to u on I does not imply uniform convergence of the sequence (u_n) on I.

Theorem 2.5 ([8, Theorem 2.1]) *Let $\{u_n(x)\} \in \omega(F)$ be a sequence of continuous functions on interval I. If $\{u_n(x)\}$ converges uniformly to a function $u(x)$ on I, then u is continuous on I.*

Theorem 2.6 ([8, Theorem 3.2]) *If the series $\sum_{k=1}^{\infty} u_k(x)$ converges uniformly on the set I and each of the terms $u_k(x)$ is continuous on I, then the sum of the series is continuous on I.*

Theorem 2.7 (Cauchy Criterion) ([8]) *A fuzzy series of functions $\sum_{k=1}^{\infty} u_k(x)$ converges uniformly on a set I if and only if for every $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}_1$ such that*

$$D\left(\sum_{k=n+1}^{m} u_k(x), \overline{0}\right) < \varepsilon \quad \textit{for all} \quad x \in I \ \textit{and for all} \quad m > n > n_0.$$

3 Completeness of the classical sets of fuzzy numbers with the level sets

Proposition 3.1 *Define D_∞ on the space $X(F)$ by*

$$\begin{aligned} D_\infty : X(F) \times X(F) &\longrightarrow \mathbb{R} \\ (u,v) &\longrightarrow D_\infty(u,v) := \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} D\left(u_k, v_k\right) = \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \sup_{\lambda\in[0,1]} d\big([u_k]_\lambda, [v_k]_\lambda\big); \end{aligned}$$

where $u = (u_k)$, $v = (v_k) \in X(F)$ and here and after X denotes any of the sets $bs(F)$, $cs(F)$ or $cs_0(F)$. Then, $(X(F), D_\infty)$ is a metric space.

Proof. Let $u = (u_k)$, $v = (v_k) \in X(F)$.

(M1) It is immediate that

$$\begin{aligned} D_\infty(u,v) = 0 \quad &\Leftrightarrow \quad \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \sup_{\lambda\in[0,1]} \left\{ d\big([u_k]_\lambda, [v_k]_\lambda\big) \right\} = 0 \\ &\Leftrightarrow \quad \sup_{n\in\mathbb{N}} \sup_{\lambda\in[0,1]} \max\left\{ \left| \sum_{k=0}^{n} (u_k)_\lambda^- - \sum_{k=0}^{n} (v_k)_\lambda^- \right|, \left| \sum_{k=0}^{n} (u_k)_\lambda^+ - \sum_{k=0}^{n} (v_k)_\lambda^+ \right| \right\} = 0 \\ &\Leftrightarrow \quad \sum_{k=0}^{n} (u_k)_\lambda^- = \sum_{k=0}^{n} (v_k)_\lambda^- \quad \text{and} \quad \sum_{k=0}^{n} (u_k)_\lambda^+ = \sum_{k=0}^{n} (v_k)_\lambda^+ \\ &\Leftrightarrow \quad \sum_{k=0}^{n} [u_k]_\lambda = \sum_{k=0}^{n} [v_k]_\lambda \Leftrightarrow u_k = v_k \Leftrightarrow u = v \end{aligned}$$

for all $\lambda \in [0,1]$.

(M2) One can easily see that

$$D_\infty(u,v)=\sup_{n\in\mathbb{N}}\sum_{k=0}^{n}D\left(u_k,v_k\right)=\sup_{n\in\mathbb{N}}\sum_{k=0}^{n}D\left(v_k,u_k\right)=D_\infty(v,u).$$

(M3) Let $u=(u_k)$, $v=(v_k)$, $w=(w_k)\in X(F)$ and by taking into account the triangle inequality and the condition $\max\{a+c,b+d\}\le\max\{a,b\}+\max\{c,d\}$ for all $a,b,c,d>0$, we observe that

$$\begin{aligned}
D_\infty(u,w) &= \sup_{n\in\mathbb{N}}\sum_{k=0}^{n}\sup_{\lambda\in[0,1]}\left\{d\big([u_k]_\lambda,[w_k]_\lambda\big)\right\}\\
&\le \sup_{n\in\mathbb{N}}\sup_{\lambda\in[0,1]}\max\left\{\left|\sum_{k=0}^{n}(u_k)_\lambda^- -\sum_{k=0}^{n}(w_k)_\lambda^-\right|,\left|\sum_{k=0}^{n}(u_k)_\lambda^+ -\sum_{k=0}^{n}(w_k)_\lambda^+\right|\right\}\\
&\le \sup_{n\in\mathbb{N}}\sup_{\lambda\in[0,1]}\max\left\{\left|\sum_{k=0}^{n}(u_k)_\lambda^- -\sum_{k=0}^{n}(v_k)_\lambda^-\right|,\left|\sum_{k=0}^{n}(u_k)_\lambda^+ -\sum_{k=0}^{n}(v_k)_\lambda^+\right|\right\}\\
&+ \sup_{n\in\mathbb{N}}\sup_{\lambda\in[0,1]}\max\left\{\left|\sum_{k=0}^{n}(v_k)_\lambda^- -\sum_{k=0}^{n}(w_k)_\lambda^-\right|,\left|\sum_{k=0}^{n}(v_k)_\lambda^+ -\sum_{k=0}^{n}(w_k)_\lambda^+\right|\right\}\\
&= D_\infty(u,v)+D_\infty(v,w),
\end{aligned}$$

where

$$\begin{aligned}
a &= \left|\sum_{k=0}^{n}(u_k)_\lambda^- -\sum_{k=0}^{n}(v_k)_\lambda^-\right|,\quad b=\left|\sum_{k=0}^{n}(u_k)_\lambda^+ -\sum_{k=0}^{n}(v_k)_\lambda^+\right|,\\
c &= \left|\sum_{k=0}^{n}(v_k)_\lambda^- -\sum_{k=0}^{n}(w_k)_\lambda^-\right|,\quad d=\left|\sum_{k=0}^{n}(v_k)_\lambda^+ -\sum_{k=0}^{n}(w_k)_\lambda^+\right|
\end{aligned}$$

for all $\lambda\in[0,1]$.

Since (M1)-(M3) are satisfied, $(X(F),D_\infty)$ is a metric space. ■

By following examples, we calculate the distance function for the spaces $bs(F)$ and $cs(F)$ with respect to the level sets.

Example 3.1 *Consider the membership functions $u_k(t)$ and $v_k(t)$ defined by the triangular fuzzy numbers as*

$$\begin{aligned}
u_k(t) &= \begin{cases} k(k+1)t-1 & , \quad \frac{1}{k(k+1)}\le t\le\frac{2}{k(k+1)},\\ 3-k(k+1)t & , \quad \frac{2}{k(k+1)}< t\le\frac{3}{k(k+1)},\\ 0 & , \quad \textit{otherwise},\end{cases}\\
v_k(t) &= \begin{cases} (k+1)^2t-1 & , \quad \frac{1}{(k+1)^2}\le t\le\frac{2}{(k+1)^2},\\ 3-(k+1)^2t & , \quad \frac{2}{(k+1)^2}< t\le\frac{3}{(k+1)^2},\\ 0 & , \quad \textit{otherwise}\end{cases}
\end{aligned}$$

for all $k \in \mathbb{N}$. It is trivial that $u_k^-(\lambda) = \frac{\lambda+1}{k(k+1)}$ and $u_k^+(\lambda) = \frac{3-\lambda}{k(k+1)}$ for all $\lambda \in [0,1]$. Therefore we see that $\sum_{k=0}^{\infty}(u_k)_\lambda^- = \lambda + 1$ and $\sum_{k=0}^{\infty}(u_k)_\lambda^+ = 3 - \lambda$. Then, it is conclude that $(u_k) \in bs(F)$. Similarly, $v_k^-(\lambda) = \frac{\lambda+1}{(k+1)^2}$ and $v_k^+(\lambda) = \frac{3-\lambda}{(k+1)^2}$ for all $\lambda \in [0,1]$. It is clear that $\sum_{k=0}^{\infty}(v_k)_\lambda^- = \frac{(\lambda+1)\pi^2}{6}$ and $\sum_{k=0}^{\infty}(v_k)_\lambda^+ = \frac{(3-\lambda)\pi^2}{6}$. Then, $(v_k) \in bs(F)$. Now we can calculate the distance between the sequences $u = (u_k)$ and $v = (v_k)$ in $bs(F)$ that

$$
\begin{aligned}
D_\infty(u,v) &= \sup_{n\in\mathbb{N}}\sum_{k=0}^{n}\sup_{\lambda\in[0,1]} d\big([u_k]_\lambda,[v_k]_\lambda\big) \\
&= \sup_{\lambda\in[0,1]}\max\left\{\sup_{n\in\mathbb{N}}\left|\sum_{k=0}^{n}(u_k)_\lambda^- - \sum_{k=0}^{n}(v_k)_\lambda^-\right|, \sup_{n\in\mathbb{N}}\left|\sum_{k=0}^{n}(u_k)_\lambda^+ - \sum_{k=0}^{n}(v_k)_\lambda^+\right|\right\} \\
&= \sup_{\lambda\in[0,1]}\max\left\{\left|\lambda+1-\frac{(\lambda+1)\pi^2}{6}\right|, \left|3-\lambda-\frac{(3-\lambda)\pi^2}{6}\right|\right\} \\
&= \sup_{\lambda\in[0,1]}\left|(3-\lambda)(1-\frac{\pi^2}{6})\right| \cong 3/2.
\end{aligned}
$$

Example 3.2 *Consider the membership functions $u_k(t)$ and $v_k(t)$ defined by the trapezoidal fuzzy numbers as*

$$
u_k(t) = \begin{cases} 2^k t - 1 & , \ \frac{1}{2^k} \le t \le \frac{2}{2^k}, \\ 1, & , \ \frac{2}{2^k} < t \le \frac{4}{2^k}, \\ 2 - 2^{k-2}t & , \ \frac{4}{2^k} < t \le \frac{8}{2^k}, \\ 0 & , \ \textit{otherwise}, \end{cases}
$$

$$
v_k(t) = \begin{cases} 6k(k+1)t - 3 & , \ \frac{1}{2k(k+1)} \le t \le \frac{2}{3k(k+1)}, \\ 1, & , \ \frac{2}{3k(k+1)} < t \le \frac{3}{4k(k+1)}, \\ 4 - 4k(k+1)t & , \ \frac{3}{4k(k+1)} < t \le \frac{1}{k(k+1)}, \\ 0 & , \ \textit{otherwise} \end{cases}
$$

for all $k \in \mathbb{N}$. It is obvious that $u_k^-(\lambda) = \frac{\lambda+1}{2^k}$ and $u_k^+(\lambda) = \frac{4(2-\lambda)}{2^k}$ for all $\lambda \in [0,1]$. Then, $\sum_{k=0}^{\infty}(u_k)_\lambda^- = 2(\lambda+1)$ and $\sum_{k=0}^{\infty}(u_k)_\lambda^+ = 8(2-\lambda)$. Similarly, $v_k^-(\lambda) = \frac{\lambda+3}{6k(k+1)}$ and $v_k^+(\lambda) = \frac{4-\lambda}{4k(k+1)}$ for all $\lambda \in [0,1]$. Then, $\sum_{k=0}^{\infty}(v_k)_\lambda^- = \frac{\lambda+3}{6}$ and $\sum_{k=0}^{\infty}(v_k)_\lambda^+ = \frac{4-\lambda}{4}$. Now, we can calculate the distance between the sequences $u = (u_k)$ and $v = (v_k)$ in $cs(F)$ that

$$
\begin{aligned}
D_\infty(u,v) &= \sup_{n\in\mathbb{N}}\sup_{\lambda\in[0,1]} d\left(\sum_{k=0}^{n}[u_k]_\lambda, \sum_{k=0}^{n}[v_k]_\lambda\right) \\
&= \sup_{\lambda\in[0,1]}\max\left\{\sup_{n\in\mathbb{N}}\left|\sum_{k=0}^{n}(u_k)_\lambda^- - \sum_{k=0}^{n}(v_k)_\lambda^-\right|, \sup_{n\in\mathbb{N}}\left|\sum_{k=0}^{n}(u_k)_\lambda^+ - \sum_{k=0}^{n}(v_k)_\lambda^+\right|\right\} \\
&= \sup_{\lambda\in[0,1]}\max\left\{\left|\frac{11\lambda+9}{6}\right|, \left|\frac{60-31\lambda}{4}\right|\right\} = \sup_{\lambda\in[0,1]}\left|\frac{60-31\lambda}{4}\right| = 15.
\end{aligned}
$$

Theorem 3.2 *The space* $(X(F), D_\infty)$ *is complete.*

Proof. Since the proof is similar for the spaces $cs(F)$ or $cs_0(F)$, we prove the theorem only for the space $bs(F)$. Let (x_m) be any Cauchy sequence in the space $bs(F)$, where $x_m = \left(x_0^{(m)}, x_1^{(m)}, x_2^{(m)}, \dots\right)$. Then, for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m, r > n_0$,

$$D_\infty(x_m, x_r) = \sup_{n\in\mathbb{N}} D\left(\sum_{k=0}^{n} x_k^{(m)}, \sum_{k=0}^{n} x_k^{(r)}\right) < \epsilon.$$

A fortiori, for every fixed $k \in \mathbb{N}$ and for $m, r > n_0$

$$D\left(\sum_{k=0}^{n} x_k^{(m)}, \sum_{k=0}^{n} x_k^{(r)}\right) < \epsilon. \tag{3.1}$$

Hence for every fixed $k \in \mathbb{N}$, by taking into account the completeness of the space (E^1, D), the sequence $(x_k^{(m)})$ is a Cauchy sequence and it converges. Now, we suppose that $\lim_{m\to\infty} x_k^{(m)} = x_k$ and $x = (x_1, x_2, \dots)$. We must show that

$$\lim_{m\to\infty} D_\infty(x_m, x) = 0 \quad \text{and} \quad x \in bs(F).$$

Letting $r \to \infty$ in (3.1), we get for all $m, n \in \mathbb{N}$ with $m > n_0$ that

$$D\left(\sum_{k=0}^{n} x_k^{(m)}, \sum_{k=0}^{n} x_k\right) < \epsilon. \tag{3.2}$$

Since the sequence $(x_k^{(m)})$ in $bs(F)$, there exists a non-negative fuzzy number M such that $D\left[\sum_{k=0}^{n} x_k^{(m)}, \overline{0}\right] < M$ for all $k \in \mathbb{N}$, where

$$M = \max\left\{\sup_{n\in\mathbb{N}} \sup_{\lambda\in[0,1]} \left|\sum_{k=0}^{n} (x_k^{(m)})_\lambda^-\right|, \sup_{n\in\mathbb{N}} \sup_{\lambda\in[0,1]} \left|\sum_{k=0}^{n} (x_k^{(m)})_\lambda^+\right|\right\}.$$

Thus, (3.2) gives together with the triangle inequality for $m > n_0$ that

$$D\left(\sum_{k=0}^{n} x_k, \overline{0}\right) \le D\left(\sum_{k=0}^{n} x_k, \sum_{k=0}^{n} x_k^{(m)}\right) + D\left(\sum_{k=0}^{n} x_k^{(m)}, \overline{0}\right) \le \epsilon + M. \tag{3.3}$$

It is clear that (3.3) holds for every fixed $k \in \mathbb{N}$ whose right-hand side does not involve k. Hence $x \in bs(F)$. Also from (3.2) we obtain for $m > n_0$ that

$$D_\infty(x_m, x) = \sup_{n\in\mathbb{N}} D\left(\sum_{k=0}^{n} x_k^{(m)}, \sum_{k=0}^{n} x_k\right) \le \epsilon.$$

This shows that $\lim_{m\to\infty} D_\infty(x_m, x) = 0$. Therefore, the space $(bs(F), D_\infty)$ is complete. ■

Proposition 3.3 $(bv(F), D_\Delta)$ *is a metric space.*

Proof. Since the metric axioms $(M1)$ and $(M2)$ are easily satisfied, we omit the detail. Let $u=(u_k)$, $v=(v_k)$, $w=(w_k)\in bv(F)$ and by taking into account the triangle inequality with the condition $\max\{a+c,b+d\}\le\max\{a,b\}+\max\{c,d\}$ for all $a,b,c,d>0$, we see that

$$\begin{aligned}
D_\Delta(u,w) &\le \sup_{\lambda\in[0,1]}\max\left\{\left|\sum_{k=0}^{\infty}(\Delta u_k)_\lambda^- - \sum_{k=0}^{\infty}(\Delta w_k)_\lambda^-\right|, \left|\sum_{k=0}^{\infty}(\Delta u_k)_\lambda^+ - \sum_{k=0}^{\infty}(\Delta w_k)_\lambda^+\right|\right\}\\
&\le \sup_{\lambda\in[0,1]}\max\left\{\left|\sum_{k=0}^{\infty}(\Delta u_k)_\lambda^- - \sum_{k=0}^{\infty}(\Delta v_k)_\lambda^-\right|, \left|\sum_{k=0}^{\infty}(\Delta u_k)_\lambda^+ - \sum_{k=0}^{\infty}(\Delta v_k)_\lambda^+\right|\right\}\\
&+ \sup_{\lambda\in[0,1]}\max\left\{\left|\sum_{k=0}^{\infty}(\Delta v_k)_\lambda^- - \sum_{k=0}^{\infty}(\Delta w_k)_\lambda^-\right|, \left|\sum_{k=0}^{\infty}(\Delta v_k)_\lambda^+ - \sum_{k=0}^{\infty}(\Delta w_k)_\lambda^+\right|\right\}\\
&= D_\Delta(u,v)+D_\Delta(v,w),
\end{aligned}$$

where

$$\begin{aligned}
a &= \left|\sum_{k=0}^{\infty}(\Delta u_k)_\lambda^- - \sum_{k=0}^{\infty}(\Delta v_k)_\lambda^-\right|, \quad b=\left|\sum_{k=0}^{\infty}(\Delta u_k)_\lambda^+ - \sum_{k=0}^{\infty}(\Delta v_k)_\lambda^+\right|,\\
c &= \left|\sum_{k=0}^{\infty}(\Delta v_k)_\lambda^- - \sum_{k=0}^{\infty}(\Delta w_k)_\lambda^-\right|, \quad d=\left|\sum_{k=0}^{\infty}(\Delta v_k)_\lambda^+ - \sum_{k=0}^{\infty}(\Delta w_k)_\lambda^+\right|.
\end{aligned}$$

Hence, triangle inequality holds. Since the metric axioms (M1)-(M3) are satisfied, $(bv(F),D_\Delta)$ is a metric space. ■

Example 3.3 *Consider the membership functions $u_k(t)$ and $v_k(t)$ defined by the triangular fuzzy numbers as*

$$u_k(t)=\begin{cases} kt-1 &, \quad \frac{1}{k}\le t\le\frac{2}{k},\\ 3-kt &, \quad \frac{2}{k}<t\le\frac{3}{k},\\ 0 &, \quad otherwise,\end{cases}$$

$$v_k(t)=\begin{cases} \frac{12^k t-3^k}{4^k-3^k} &, \quad \frac{1}{4^k}\le t\le\frac{1}{3^k},\\ \frac{3^k-6^k t}{3^k-2^k} &, \quad \frac{1}{3^k}<t\le\frac{1}{2^k},\\ 0 &, \quad otherwise\end{cases}$$

for all $k\in\mathbb{N}$. Then, $u_k^-(\lambda)=\frac{\lambda+1}{k}$ and $u_k^+(\lambda)=\frac{3-\lambda}{k}$. Additionally, since

$$[(\Delta u)_k]_\lambda=[(u_k)_\lambda^- - (u_{k+1})_\lambda^-,(u_k)_\lambda^+ - (u_{k+1})_\lambda^+]=\left[\frac{\lambda+1}{k(k+1)},\frac{3-\lambda}{k(k+1)}\right],$$

$(u_k)\in bv(F)$. Similarly, $v_k^-(\lambda)=\frac{\lambda}{3^k}-\frac{\lambda-1}{4^k}$ and $v_k^+(\lambda)=\frac{\lambda}{3^k}+\frac{1-\lambda}{2^k}$ then,

$$[(\Delta v)_k]_\lambda=[(v_k)_\lambda^- - (v_{k+1})_\lambda^-,(v_k)_\lambda^+ - (v_{k+1})_\lambda^+]=\left[\frac{2\lambda}{3^{k+1}}-\frac{3(\lambda-1)}{4^{k+1}},\frac{2\lambda}{3^{k+1}}+\frac{1-\lambda}{2^{k+1}}\right]$$

and $(v_k) \in bv(F)$. *Now we can calculate that*

$$\begin{aligned} D(u,v) &= \sum_{k=0}^{\infty} D\left[(\Delta u)_k, (\Delta v)_k\right] = \sup_{\lambda\in[0,1]} \sum_{k=0}^{\infty} d\left\{[(\Delta u)_k]_\lambda, [(\Delta v)_k]_\lambda\right\} \\ &= \sup_{\lambda\in[0,1]} \max\left\{\left|\sum_{k=0}^{\infty}[(\Delta u)_k]_\lambda^- - \sum_{k=0}^{\infty}[(\Delta v)_k]_\lambda^-\right|, \left|\sum_{k=0}^{\infty}[(\Delta u)_k]_\lambda^+ - \sum_{k=0}^{\infty}[(\Delta v)_k]_\lambda^+\right|\right\} \\ &= \sup_{\lambda\in[0,1]} \max\left\{\left|\sum_{k=0}^{\infty}\frac{\lambda+1}{k(k+1)} - \sum_{k=0}^{\infty}\left[\frac{2\lambda}{3^{k+1}} - \frac{3(\lambda-1)}{4^{k+1}}\right]\right|,\right. \\ &\quad \left.\left|\sum_{k=0}^{\infty}\frac{3-\lambda}{k(k+1)} - \sum_{k=0}^{\infty}\left(\frac{2\lambda}{3^{k+1}} + \frac{1-\lambda}{2^{k+1}}\right)\right|\right\} \\ &= \sup_{\lambda\in[0,1]} \max\{|\lambda|, |2-\lambda|\} = \sup_{\lambda\in[0,1]} \{2-\lambda\} = 2. \end{aligned}$$

Theorem 3.4 $(bv(F), D_\Delta)$ *is a complete metric space.*

Proof. Let (u_m) be any Cauchy sequence in the space $bv(F)$, where $u_m = \left\{u_0^{(m)}, u_1^{(m)}, u_2^{(m)}, \cdots\right\}$ for all $m \in \mathbb{N}$. Then, for every $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that for all $m, r > N(\varepsilon)$,

$$D_\Delta(u_m, u_r) = \sum_{k=0}^{\infty} D\left[(\Delta u)_k^{(m)}, (\Delta u)_k^{(r)}\right] < \varepsilon.$$

A fortiori, for every fixed $k \in \mathbb{N}$ and for $m, r > N(\varepsilon)$

$$D\left[(\Delta u)_k^{(m)}, (\Delta u)_k^{(r)}\right] < \varepsilon. \tag{3.4}$$

Hence for every fixed $k \in \mathbb{N}$, by taking into account the completeness of the space (E^1, D), the sequence $\{(\Delta u)_k^{(m)}\}$ is a Cauchy sequence and it converges. Now, we suppose that $\lim_{m\to\infty} (\Delta u_k)^{(m)} = (\Delta u)_k$, $\Delta u = \{(\Delta u)_0, (\Delta u)_1, \cdots, (\Delta u)_k, \cdots\}$. We must show that

$$\lim_{m\to\infty} D_\Delta(u_m, u) = 0 \quad \text{and} \quad u = (u_k) \in bv(F).$$

We have from (3.4) for each $j \in \mathbb{N}$ and $m, r > N(\varepsilon)$ that

$$\sum_{k=0}^{j} D\left[(\Delta u)_k^{(m)}, (\Delta u)_k^{(r)}\right] \leq D_\Delta(u_m, u_r) < \varepsilon. \tag{3.5}$$

Take any $m > N(\varepsilon)$. Let us pass to limit firstly $r \to \infty$ and next $j \to \infty$ in (3.5) to obtain $D_\Delta(u_m, u) \leq \varepsilon$. Since the sequence (u_m) is in $bv(F)$, there exists a non-negative fuzzy number M such that $D_\Delta(u_m, \overline{0}) < M$ for all $m \in \mathbb{N}$, where

$$M = \sup_{\lambda\in[0,1]} \max\left\{\left|\sum_{k=0}^{\infty}((\Delta u)_k^{(m)})_\lambda^-\right|, \left|\sum_{k=0}^{\infty}((\Delta u)_k^{(m)})_\lambda^+\right|\right\}.$$

By using inclusion (3.5) and Minkowski's inequality for each $j \in \mathbb{N}$ that

$$\sum_{k=0}^{j} D[(\Delta u)_k, \overline{0}] \leq D_{\Delta}(u_m, u) + D_{\Delta}(u_m, \overline{0}) \leq \varepsilon + M$$

which implies that $u \in bv(F)$. Since $D_{\Delta}(u_m, u) \leq \epsilon$ for all $m > N(\varepsilon)$ it follows that $\lim_{m\to\infty} D_{\Delta}(u_m, u) = 0$. Since (u_m) is an arbitrary Cauchy sequence, the space $(bv(F), D_{\Delta})$ is complete.

This step completes the proof. ■

Theorem 3.5 ([12]) *Define the relation D_p on the space $bv_p(F)$ by*

$$\begin{aligned} D_p : bv_p(F) \times bv_p(F) &\longrightarrow \mathbb{R} \\ (u, v) &\longrightarrow D_p(u, v) = \left\{ \sum_{k=0}^{\infty} D\left[(\Delta u)_k, (\Delta v)_k\right]^p \right\}^{1/p}, \quad (1 \leq p < \infty), \end{aligned}$$

where $u = (u_k)$ and $v = (v_k) \in bv_p(F)$ and the difference sequence $(\Delta u)_k = u_k - u_{k-1}$, $(u_{-1} = \overline{0})$ for all $k \in \mathbb{N}$. Then, $(bv_p(F), D_p)$ is a metric space.

Theorem 3.6 ([12, Theorem 7]) $(bv_p(F), D_{\Delta})$ *is a complete metric space.*

4 The duals of the classical sets of sequences of fuzzy numbers with the level sets

Following Başar [3], first we define the alpha-, beta- and gamma-duals of a set $\mu(F) \subset \omega(F)$ which are respectively denoted by $\{\mu(F)\}^{\alpha}$, $\{\mu(F)\}^{\beta}$ and $\{\mu(F)\}^{\gamma}$, as follows:

$$\begin{aligned} \{\mu(F)\}^{\alpha} &:= \Big\{ u = (u_k) \in \omega(F) : (u_k v_k) \in \ell_1(F) \ \text{ for all } \ (v_k) \in \mu(F) \Big\}, \\ \{\mu(F)\}^{\beta} &:= \Big\{ u = (u_k) \in \omega(F) : (u_k v_k) \in cs(F) \ \text{ for all } \ (v_k) \in \mu(F) \Big\}, \\ \{\mu(F)\}^{\gamma} &:= \Big\{ u = (u_k) \in \omega(F) : (u_k v_k) \in bs(F) \ \text{ for all } \ (v_k) \in \mu(F) \Big\}. \end{aligned}$$

Following Talo and Başar [10], we give the classical sets $\ell_{\infty}(F)$, $c(F)$, $c_0(F)$, $\ell_p(F)$, $bv_p(F)$ and $bv_{\infty}(F)$ consisting of the bounded, convergent, null, absolutely p-summable, p-bounded variation and bounded difference sequences of fuzzy numbers, i.e.,

$$
\begin{aligned}
\ell_\infty(F) &:= \left\{(u_k) \in \omega(F) : \sup_{k\in\mathbb{N}} D(u_k,\overline{0}) < \infty\right\},\\
c(F) &:= \left\{(u_k) \in \omega(F) : \exists l \in E^1 \ni \lim_{k\to\infty} D(u_k,l) = 0\right\},\\
c_0(F) &:= \left\{(u_k) \in \omega(F) : \lim_{k\to\infty} D(u_k,\overline{0}) = 0\right\},\\
\ell_p(F) &:= \left\{(u_k) \in \omega(F) : \sum_{k=0}^{\infty} D(u_k,\overline{0})^p < \infty\right\},\\
bv_p(F) &:= \left\{u = (u_k) \in \omega(F) : \sum_{k=0}^{\infty} \left\{D\left(\Delta u_k,\overline{0}\right)\right\}^p < \infty\right\},\\
bv_\infty(F) &:= \left\{u = (u_k) \in \omega(F) : \sup_{k\in\mathbb{N}} D\left(\Delta u_k,\overline{0}\right) < \infty\right\},
\end{aligned}
$$

where the difference sequence $\Delta u = (\Delta u_k)$ by $\Delta u_k = u_k - u_{k-1}$, $(u_{-1} = \overline{0})$ for all $k \in \mathbb{N}$ and $1 \leq p < \infty$. Additionally, the space $bv_0(F)$ is the intersection of the spaces $bv(F)$ and $c_0(F)$.

Lemma 4.1 ([10, Lemma 3.1]) *Let d denotes the set of all absolutely convergent series of fuzzy numbers, i.e.,*

$$d := \left\{u = (u_k) \in \omega(F) : \sum_{k=0}^{\infty} D(u_k,\overline{0}) < \infty\right\}.$$

Then, the set d is identical to the set $\ell_1(F)$.

Theorem 4.2 ([10, Theorem 3.2]) *The $\alpha-$dual of the sets $c(F)$, $c_0(F)$ and $\ell_\infty(F)$ of sequences of fuzzy numbers is the set $\ell_1(F)$.*

Definition 4.1 ([10, Definition 3.3]) *A set $\mu(F) \subset \omega(F)$ is said to be solid if $(v_k) \in \mu(F)$ whenever $D(v_k,\overline{0}) \leq D(u_k,\overline{0})$ for all $k \in \mathbb{N}$ for some $(u_k) \in \mu(F)$. It is known that the alpha- and gamma-duals of a set of sequences of fuzzy numbers are identical if it is solid.*

Lemma 4.3 *The following statements hold:*

(a) ([10, Theorem 3.4]) *The sets $c_0(F)$, $\ell_p(F)$ and $\ell_\infty(F)$ are solid.*

(b) ([10, Theorem 3.5]) *The beta-dual of the sets $c(F)$, $c_0(F)$ and $\ell_\infty(F)$ is the set $\ell_1(F)$.*

(c) ([10, Theorem 3.6]) *The alpha- and beta-duals of the set $\ell_p(F)$ are the set $\ell_q(F)$, where $1 \leq p < \infty$.*

(d) ([10, Corollary 3.7]) *The gamma-dual of the sets* $\ell_p(F)$, $c_0(F)$ *and* $\ell_\infty(F)$ *is the set* $\ell_1(F)$, *where* $0 < p < 1$.

(e) ([10, Lemma 2.6]) *We have*

(i) $D(uv, \overline{0}) \leq D(u, \overline{0})D(v, \overline{0})$ *for all* $u, v \in E^1$,

(ii) *If* $u_k \to u$, *as* $k \to \infty$ *then* $D(u_k, \overline{0}) \to D(u, \overline{0})$, *as* $k \to \infty$; *where* $(u_k) \in \omega(F)$.

Theorem 4.4 (Weierstrass M test) ([10]) *Let* $u_k : [a, b] \to \mathbb{R}$ *be given. If there exists an* $M_k > 0$ *such that* $|u_k(x)| \leq M_k$ *for all* $k \in \mathbb{N}$ *and* $\sum_k M_k$ *converges, then* $\sum_k u_k(x)$ *is uniformly and absolutely convergent in* $[a, b]$.

Lemma 4.5 ([10, Theorem 2.18]) *If the series* $\sum_k u_k$ *and* $\sum_k v_k$ *converge, then* $D\big(\sum_k u_k, \sum_k v_k\big) \leq \sum_k D(u_k, v_k)$.

Lemma 4.6 (The Comparison test) ([18, Theorem 4.4]) *Let* (u_k) *and* (v_k) *be the sequences of non-negative fuzzy numbers and* $u_k \preceq v_k$ *for all* $k \in \mathbb{N}$. *Then, the following statements hold:*

(i) *If* $\sum_{k=0}^{\infty} v_k$ *converges, then* $\sum_{k=0}^{\infty} u_k$ *converges.*

(ii) *If* $\sum_{k=0}^{\infty} u_k$ *diverges, then* $\sum_{k=0}^{\infty} v_k$ *diverges.*

(iii) *The absolute convergence of a series of fuzzy numbers implies the convergence of the series.*

Now, we give the alpha-, beta- and gamma-duals of the sets $bs(F)$, $cs(F)$, $cs_0(F)$ and $bv(F)$.

Theorem 4.7 *The alpha-dual of the sets* $cs(F)$, $bs(F)$, $bv_1(F)$ *and* $bv_0(F)$ *is the set* $\ell_1(F)$.

Proof. We prove the case $\{cs(F)\}^\alpha = \ell_1(F)$ and the rest can be obtained similarly.

Let $u = (u_k) \in \ell_1(F)$. Then, $\sum_{k=0}^{\infty} D(u_k, \overline{0})$ converges. Therefore, we derive by using the fact given in Part (i) of Lemma 4.6 that

$$\sum_{k=0}^{\infty} D(u_k v_k, \overline{0}) \leq \sum_{k=0}^{\infty} D(u_k, \overline{0})D(v_k, \overline{0}) \leq M \sum_{k=0}^{\infty} D(u_k, \overline{0}).$$

If we take $v = (v_k) \in cs_0(F) \subset cs(F)$, then we have $\sum_{k=0}^{\infty} D(u_k v_k, \overline{0}) < \infty$ which gives that $\ell_1(F) \subseteq \{cs(F)\}^\alpha$.

Conversely, suppose that $u = (u_k) \in \{cs(F)\}^{\alpha}$ and $v = (v_k) \in cs_0(F)$. Then, the series $\sum_{k=0}^{\infty} D(v_k, \overline{0})$ converges. Since

$$\begin{aligned} |(u_k v_k)_{\lambda}^{-}| &\leq \sum_{k=0}^{\infty} D(u_k v_k, \overline{0}) \leq \sum_{k=0}^{\infty} D(u_k, \overline{0}) D(v_k, \overline{0}) \\ |(u_k v_k)_{\lambda}^{+}| &\leq \sum_{k=0}^{\infty} D(u_k v_k, \overline{0}) \leq \sum_{k=0}^{\infty} D(u_k, \overline{0}) D(v_k, \overline{0}), \end{aligned}$$

Weierstrass' M Test yields that $\sum_{k=0}^{\infty} (u_k v_k)_{\lambda}^{-}$ and $\sum_{k=0}^{\infty} (u_k v_k)_{\lambda}^{+}$ converge uniformly and hence $\sum_{k=0}^{\infty} u_k v_k$ converges whenever $\sum_{k=0}^{\infty} D(u_k, \overline{0})$ converges. Therefore, we have $\{cs(F)\}^{\alpha} \subseteq \ell_1(F)$.

This step concludes the proof. ■

Theorem 4.8 *The following statements hold:*

(i) $\{cs(F)\}^{\beta} = bv_1(F)$.

(ii) $\{bv(F)\}^{\beta} = cs(F)$.

(iii) $\{bv_0(F)\}^{\beta} = bs(F)$.

(iv) $\{bs(F)\}^{\beta} = bv_0(F)$.

Proof. Since the other parts can be similarly proved, we consider only Part (i).

Let $u = (u_k) \in \{cs(F)\}^{\beta}$ and $w = (w_k) \in c_0(F)$. Define the sequence $v = (v_k) \in cs(F)$ by $v_k = w_k - w_{k+1}$ for all $k \in \mathbb{N}$. Therefore, $\sum_{k=0}^{\infty} u_k v_k$ converges, but

$$\sum_{k=0}^{n} (w_k - w_{k+1}) u_k = \sum_{k=0}^{n-1} w_k (u_k - u_{k-1}) - w_{n+1} u_n \tag{4.1}$$

and the inclusion $\ell_1(F) \subset cs(F)$ yields that $(u_k) \in \{cs(F)\}^{\beta} \subset \{\ell_1(F)\}^{\beta} = \ell_{\infty}(F)$. Then we derive by passing to limit in (4.1) as $n \to \infty$ which implies that

$$\sum_{k=0}^{\infty} (w_k - w_{k+1}) u_k = \sum_{k=0}^{\infty} w_k (u_k - u_{k-1})$$

for every $k \in \mathbb{N}$. Hence $(u_k - u_{k-1}) \in \{c_0(F)\}^{\beta} = \{c_0(F)\}^{\alpha} = \ell_1(F)$, i.e., $u \in bv_1(F)$. Therefore, $\{cs(F)\}^{\beta} \subseteq bv_1(F)$.

Conversely, suppose that $u = (u_k) \in bv_1(F)$. Then, $(u_k - u_{k-1}) \in \ell_1(F)$. Further, if $v = (v_k) \in cs(F)$, the sequence (w_n) defined by $w_n = \sum_{k=0}^{n} v_k$ for all $n \in \mathbb{N}$, is an element of the space $c(F)$. Since $\{c(F)\}^{\alpha} = \ell_1(F)$, the series $\sum_{k=0}^{\infty} w_k(u_k - u_{k+1})$ is absolutely convergent. Also, we have

$$\left|\sum_{k=m}^{n}(w_k - w_{k-1})u_k\right| \leq \left|\sum_{k=m}^{n-1} w_k(u_k - u_{k+1})\right| + \left|w_n u_n - w_{m-1}u_m\right|. \tag{4.2}$$

Since $(w_n) \in c(F)$ and $(u_k) \in bv_1(F) \subset c(F)$, the right-hand side of inequality (4.2) converges to zero as $m, n \to \infty$. Hence, the series $\sum_{k=0}^{\infty}(w_k - w_{k-1})u_k$ or $\sum_{k=0}^{\infty} u_k v_k$ converges and so, $bv_1(F) \subseteq \{cs(F)\}^{\beta}$. Thus, $\{cs(F)\}^{\beta} = bv_1(F)$. ■

Theorem 4.9 *The following statements hold:*

(i) $\{cs(F)\}^{\gamma} = \{bs(F)\}^{\gamma} = bv_1(F)$.

(ii) $\{bv_0(F)\}^{\gamma} = \{bv_1(F)\}^{\gamma} = bs(F)$.

Proof. We prove only Part (i) for $\{cs(F)\}^{\gamma}$, the rest can be proved in a similar way.

By Theorem 4.8, we have $bv_1(F) \subseteq \{cs(F)\}^{\beta}$ and since $\{cs(F)\}^{\beta} \subset \{cs(F)\}^{\gamma}$, so $bv_1(F) \subset \{cs(F)\}^{\gamma}$. We need to show that $\{cs(F)\}^{\gamma} \subset bv_1(F)$. Let $u = (u_n) \in \{cs(F)\}^{\gamma}$ and $v = (v_n) \in c_0(F)$. Then, for the sequence $(w_n) \in cs(F)$ defined by $w_n = v_n - v_{n+1}$ for all $n \in \mathbb{N}$, we can find a fuzzy constant $K \succ 0$ such that $\left|\sum_{k=0}^{n} u_k w_k\right| \leq K$ for all $n \in \mathbb{N}$. Since $(v_n) \in c_0(F)$ and $(u_n) \in \{cs(F)\}^{\gamma} \subset \ell_\infty(F)$, there exists a fuzzy constant $M \succ 0$ such that $|u_n v_n| \leq M$ for all $n \in \mathbb{N}$. Therefore,

$$\left|\sum_{k=0}^{n}(u_k - u_{k-1})v_k\right| \leq \left|\sum_{k=0}^{n+1} u_k(v_k - v_{k+1})\right| + \left|v_{n+2}u_{n+1}\right| \leq K + M.$$

Hence $(u_k - u_{k-1}) \in \{c_0(F)\}^{\gamma} = \{c_0(F)\}^{\alpha} = \ell_1(F)$ from Part (d) of Lemma 4.3, i.e., $(u_n) \in bv_1(F)$. Therefore, since the inclusion $\{cs(F)\}^{\gamma} \subset bv_1(F)$ holds, we conclude that $\{cs(F)\}^{\gamma} = bv_1(F)$, as desired. ■

Now, we give an example related to the alpha-, beta- and gamma- duals of the set $cs(F)$.

Example 4.1 *Consider the triangular fuzzy numbers with the membership functions u_k and v_k defined by*

$$u_k(t) = \begin{cases} k(k+1)t - 1 & , \quad \frac{1}{k(k+1)} \leq t \leq \frac{2}{k(k+1)}, \\ 3 - k(k+1)t & , \quad \frac{2}{k(k+1)} < t \leq \frac{3}{k(k+1)}, \\ 0 & , \quad otherwise, \end{cases}$$

$$v_k(t) = \begin{cases} (2k-1)(2k+1)t - 1 & , \quad \frac{1}{(2k-1)(2k+1)} \leq t \leq \frac{2}{(2k-1)(2k+1)}, \\ 8 - 2t(2k-1)(2k+1) & , \quad \frac{2}{(2k-1)(2k+1)} < t \leq \frac{4}{(2k-1)(2k+1)}, \\ 0 & , \quad otherwise \end{cases}$$

for all $k \in \mathbb{N}$. Since $u_k^-(\lambda) = \frac{\lambda+1}{k(k+1)}$ and $u_k^+(\lambda) = \frac{3-\lambda}{k(k+1)}$, $(u_k) \in \ell_1(F)$. Similarly, $v_k^-(\lambda) = \frac{\lambda+1}{(2k-1)(2k+1)}$ and $v_k^+(\lambda) = \frac{8-\lambda}{2(2k-1)(2k+1)}$ then, $\sum\limits_{k=0}^{\infty} v_k = \left[\frac{\lambda+1}{2}, \frac{8-\lambda}{4}\right] \in cs(F)$. Therefore, a straightforward calculation yields that

$$\begin{aligned}
\sum_{k=0}^{\infty}(u_k v_k)_\lambda^- &= \sum_{k=0}^{\infty}\min\left\{(u_k)_\lambda^-(v_k)_\lambda^-, (u_k)_\lambda^-(v_k)_\lambda^+, (u_k)_\lambda^+(v_k)_\lambda^-, (u_k)_\lambda^+(v_k)_\lambda^+\right\} \\
&= \sum_{k=0}^{\infty}\min\left\{\frac{(\lambda+1)^2}{k(k+1)(2k-1)(2k+1)}, \frac{(\lambda+1)(8-\lambda)}{2k(k+1)(2k-1)(2k+1)},\right. \\
&\qquad \left.\frac{(3-\lambda)(\lambda+1)}{2k(k+1)(2k-1)(2k+1)}, \frac{(3-\lambda)(8-\lambda)}{2k(k+1)(2k-1)(2k+1)}\right\} \\
&= \sum_{k=0}^{\infty}\frac{(\lambda+1)^2}{k(k+1)(4k^2-1)}. \qquad (4.3)
\end{aligned}$$

Then, we see for the sequences $(u_k) = \left\{\frac{(\lambda+1)^2}{k(k+1)(2k-1)(2k+1)}\right\}$ and $(v_k) = \left\{\frac{(\lambda+1)^2}{(2k-1)(2k+1)}\right\}$ of fuzzy numbers that $u_k \preceq v_k$ for all $k \in \mathbb{N}$. Since the series $\sum\limits_{k=0}^{\infty} v_k$ converges, then the series (4.3) also converges by using Comparison Test in Part (i) of Lemma 4.6. Similarly,

$$\sum_{k=0}^{\infty}(u_k v_k)_\lambda^+ = \sum_{k=0}^{\infty}\frac{(3-\lambda)(8-\lambda)}{2k(k+1)(4k^2-1)}. \qquad (4.4)$$

Taking the sequences $(u_k) = \left\{\frac{(3-\lambda)(8-\lambda)}{2k(k+1)(2k-1)(2k+1)}\right\}$ and $(v_k) = \left\{\frac{(3-\lambda)(8-\lambda)}{(2k-1)(2k+1)}\right\}$ of fuzzy numbers, we have $u_k \preceq v_k$ for all $k \in \mathbb{N}$. Since the series $\sum\limits_{k=0}^{\infty} v_k$ converges, then series (4.4) also converges. Therefore,

$$\begin{aligned}
\sum_{k=0}^{\infty} D(u_k v_k, \overline{0}) &= \sup_{\lambda\in[0,1]}\max\left\{\left|\sum_{k=0}^{\infty}(u_k v_k)_\lambda^-\right|, \left|\sum_{k=0}^{\infty}(u_k v_k)_\lambda^+\right|\right\} \\
&= \sup_{\lambda\in[0,1]}\max\left\{\left|\sum_{k=0}^{\infty}\frac{(\lambda+1)^2}{k(k+1)(4k^2-1)}\right|, \left|\sum_{k=0}^{\infty}\frac{(3-\lambda)(8-\lambda)}{2k(k+1)(4k^2-1)}\right|\right\} < \infty.
\end{aligned}$$

Hence, $(u_k v_k) \in \ell_1(F) = \{cs(F)\}^\alpha$. Additionally, by using $\lim\limits_n \sum_{k=0}^n D\left(u_k v_k, \overline{l}\right) = 0$ for some $\overline{l} \in E^1$, one can see that $(u_k v_k) \in cs(F)$. It is obvious that $(u_k) \in bv_1(F) = \{cs(F)\}^\beta = \{cs(F)\}^\gamma$. Indeed

$$\begin{aligned}
\sum_{k=0}^{\infty} D\left[\Delta(u_k), \overline{0}\right] &= \sup_{\lambda\in[0,1]}\sum_{k=0}^{\infty} d\big([u_k - u_{k-1}]_\lambda, \overline{0}\big) \\
&= \sup_{\lambda\in[0,1]}\max\left\{\left|\sum_{k=0}^{\infty}(u_k)_\lambda^- - (u_{k-1})_\lambda^-\right|, \left|\sum_{k=0}^{\infty}(u_k)_\lambda^+ - (u_{k-1})_\lambda^+\right|\right\} \\
&= \sup_{\lambda\in[0,1]}\max\left\{\left|\sum_{k=0}^{\infty}\left[\frac{\lambda+1}{k(k+1)} - \frac{\lambda+1}{k(k-1)}\right]\right|, \left|\sum_{k=0}^{\infty}\left(\frac{3-\lambda}{k(k+1)} - \frac{3-\lambda}{k(k-1)}\right)\right|\right\} \\
&= \sup_{\lambda\in[0,1]}\max\left\{\sum_{k=0}^{\infty}\left[\frac{2(\lambda+1)}{k(k+1)(k-1)}\right], \sum_{k=0}^{\infty}\left[\frac{2(3-\lambda)}{k(k+1)(k-1)}\right]\right\} < \infty.
\end{aligned}$$

Therefore, $(u_k) \in bv_1(F)$.

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