ACOUSTIC WAVE MODELLING USING TWO DIFFERENT NUMERICAL METHODS

Murat Sarı*, İsmail Demir**

*Pamukkale University, Faculty of Art and Science, Division of Mathematics, Kınıklı/Denizli
**Çanakkale Onsekizmart University, Faculty of Art and Science, Division of Mathematics, Çanakkale

This paper analyses various 2D acoustic wave propagation problems in the time domain BEM through geophysical environments. To this end the existing BEM code for the boundary nodes is expanded to optional internal nodes. Using appropriate and predominant temporal variations for the field quantities the time-related kernels are obtained explicitly. The BEM and FDM solutions presented are generated using synthetic seismograms and are seen to be stable. The qualitative agreement between the two methods is excellent.

Key Words : BEM, Acoustic wave propagation, Geophysical structure, Synthetic seismogram

İKİ FARKLI NÜMERİK METOT KULLANARAK AKUSTİK DALGA MODELLEMESİ

Bu makale, jeofiziksel ortamlardan geçen çeşitli iki boyutlu akustik dalga yayılımı problemlerini zaman-domain SEM yardımıyla analiz eder. Bu, sınır noktaları için mevcut olan SEM programlarının keyfi sayıdaki iç noktalar için genişletilmesi ile başarılır. Alan değişkenleri için uygun ve sıklıkla kullanılan temporal değişimlerin benimsenmesi ile zamana bağlı kernellar açık olarak elde edilir. Sunulan SEM ve SFM sonuçları sentetik sismogramların kullanılmasıyla genelleştirilir ve bu sonuçların kararlı olduğu görülür. İki metot sonuçları arasındaki niteliksel uyum mükemmeldir.

Anahtar Kelimeler : SEM, Akustik dalga yayılımı, Jeofiziksel yapı, Sentetik sismogram

1. INTRODUCTION

Due to difficulties of obtaining an analytic solution to a dynamic problem, and because of advances in computer technology, numerical methods have become more and more popular over the last three decades. The finite element method (FEM) is probably the most popular and well studied to solve dynamic problems see for example (Zienkiewicz, 1977). However the FEM, like the finite difference method (FDM), requires full-discretization of the domain and in case of an infinite or semi-infinite domain a complete discretization is not practical. Another numerical technique called the boundary element method (BEM) only requires the discretization of the boundaries. Since the BEM has natural advantage over the domain approaches, the BEM has been used over the last three decades to tackle many different problems in various disciplines (Brebbia and Dominguez, 1992; Beskos, 1997). In this paper the BEM is used to ascertain the behaviour of acoustic waves governed by the scalar wave equation through single-layered bounded and unbounded media.

The general time domain BEM formulation for the scalar wave equation was established by Mansur (1983). Since then many BEM formulations have been employed to analyse various problems in the time domain for example (Israil and Banerjee, 1990; Wang and Takemiya, 1992; Carrer and Mansur, 1994).

Accuracy and numerical stability of the BEM solutions was discussed in (Sarı, 2000; Meijs et al., 1989). So far in the BEM community, attentions were usually paid to singular integrals, variation of the field variables, use of time or frequency methods and lately stability of the BEMs. The minuses and pluses of the earlier works were

examined by Birgisson and Crouch, (1998) in the context of the 2D-elastodynamic problems.

This work applies the best and commonest (Gallego and Dominguez, 1996; Richter, 1997; Bonnet, 1998) scheme, that is constant and linear time variations for the flux and potential, respectively. For the space approximation of the field variables constant elements are used.

Physically, the paper focuses on the determination of the wave behaviour in homogeneous media of finite and infinite extent. Despite increased volume of research devoted to the analysis of wave propagation problems, most authors use the FDM to do their investigation. However, there is some literature that uses the BEM. Some of the few examples are (Ahmad and Banerjee, 1988; Cheung et al., 1993). Note that their investigations are not in the time domain.

Consideration of the wave propagation problems in the BEM is a deep problem due to its structural complexity. The equation is solved for geophysical structures of finite and infinite extent. Synthetic seismograms generated from the solutions are presented. The sensitivity of the solutions to varying time steps is reported in Sari (2000). To solve our problem, a main program of Dominguez (1993) is expanded to internal potential variables. The present analysis shows that the BEM is capable of treating large-sized problems. A number of examples are presented and comparisons with the FDM results are also made. The FDM formulation can be found in (Demir, 1999).

2. THE INTEGRAL EQUATION

The 2D governing wave equation corresponding to a homogeneous isotropic elastic body Ω enclosed by the boundary B is given by Morse and Feshbach, (1953), as follows:

$$c^2\phi_{,ii} + f = \ddot{\phi} \tag{1}$$

In this equation ϕ, f and $\ddot{\phi}$ are functions of position and time, and represent potential, body source and acceleration respectively, whilst c is the wave speed. In the above $\phi_{,ii}$ and $\ddot{\phi}$ are the second derivatives of the potential ϕ with respect to the direction x_i and the time t, respectively.

For all points x of the boundary B with time $t \in \mathbb{R}$, the boundary conditions may be specified

conveniently using the two known functions $\bar{\phi}(x,t)$, $\bar{q}(x,t)$ defined by:

$$\phi(x,t) = \begin{cases} \bar{\phi}(x,t) & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases} \tag{2}$$

where $x \in B_2 \subseteq B$, and

$$q(x,t) = \frac{\partial\phi(x,t)}{\partial n} = \begin{cases} \bar{q}(x,t) & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases} \tag{3}$$

with $\mathbf{x} \in B_1 \subseteq B$. Here, n is the outward unit normal vector at the position vector x, B_1 and B_2 are parts of boundary $B = B_1 + B_2$ and where $B_1 \cap B_2 = \varnothing$. Considering the domain Ω bounded by its boundary *B*; the fundamental solutions and the actual states of the governing differential equation (1) can be combined through the use of the dynamical reciprocity theorem, to give the following time-domain potential boundary integral equation,

$$
\alpha^i\phi(y^i,t) = \int_0^{t^+}\!\!\int_B \phi^s(x,t^*;y^i)q(x,\tau)dBd\tau \\
- \int_0^{t^+}\!\!\int_B q^s(x,t^*;y^i)\phi(x,\tau)dBd\tau \tag{4}
$$

Where $t^* = t - \tau$. Equation (4) expresses the scalar potential field at any point of a medium as a function the field quantities on the boundary. Here α^i depends only upon the local geometry of the body at the load point y^i. In equation (4), zero body source and zero initial conditions are assumed. Also in the equation:

$$
\phi^s(x,t^*;y^i) = \frac{c}{2\pi(c^2t^{*2}-r^2)^{1/2}} \quad \text{if } ct^* > r \\
= 0 \qquad\qquad \text{otherwise} \tag{5}
$$

$$
q^s = \frac{\partial\phi^s(x,t^*;y^i)}{\partial n} = \frac{\partial r}{\partial n}\frac{cH(ct^*-r)}{2\pi(c^2t^{*2}-r^2)^{3/2}} \tag{6}
$$

Where *r* is the distance between the load point y^i and the field point x. In the above ϕ and ϕ^s represent the actual and fundamental solution states of the scalar potential respectively, whilst *H* stands for the Heaviside function. As can be seen from the

fundamental solutions (5) and (6) the disturbance sent initially from the load point y^i is received at the field point x at time $r < ct^*$ and decays as t^* increases. Here the upper limit t^+ is used to avoid ending the integration at the peak of the Dirac delta function (Morse and Feshbach, 1953).

Since the radiation conditions (Eringen and Şuhubi, 1975) are automatically satisfied, the boundary integral equation (4) is valid for unbounded media as well as bounded media. These conditions ensure the uniqueness of the BEM solution (Bonnet, 1998). It may, therefore, be seen that there is no need to discretize an external boundary when it is of infinite extent. This makes the BEM advantageous over the domain techniques. The 2D fundamental solutions satisfy the causality, reciprocity and time translation properties.

3. NUMERICAL FORMULATION

In this work, linear line elements are used to approximate the boundary. Temporal variation of the kernels of equation (4) is used to obtain a numerical solution of the partial differential equation. The evaluation of the kernels is discussed in Sari (2000). In the corresponding work, the potential and its normal derivative are interpolated by linear and constant approximations in time, respectively. The choice of the time functions is also discussed in the previously mentioned work as well as the research of (Gallego and Dominguez, 1996; Birgisson and Crouch, 1998; Mansur et al. (1998).

In this analysis time is divided into *n* equal intervals, $t = n\Delta t$. The approximations of the field quantities are:

$$\phi(x,\tau) = \sum_m \eta^m(\tau)\phi^m(x)$$

$$q(x,\tau) = \sum_m \mu^m(\tau)q^m(x) \tag{7}$$

where $\eta^m(\tau)$ and $\mu^m(\tau)$ are temporal interpolation functions. Moreover ϕ^m and q^m indicate the potential and flux, respectively, at time $t_m = m\Delta t$ at point x. The time interpolation functions used by us explicitly are:

$$\eta^m(\tau) = \begin{cases} \dfrac{(\tau - \tau_{m-1})}{\Delta t} & \text{if } \tau_{m-1} < \tau < \tau_m \\[2mm] \dfrac{(\tau_{m+1} - \tau)}{\Delta t} & \text{if } \tau_m < \tau < \tau_{m+1} \\[2mm] 0 & \text{else} \end{cases} \tag{8}$$

$$\mu^m(\tau) = \begin{cases} 1 & \text{if } \tau_{m-1} < \tau < \tau_m \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

The discretized form of equation (4) is:

$$\alpha^i \phi^{ni} = \sum_{m=1}^n [\int_B U^{nm} q^m(x) dB]$$
$$- \sum_{m=1}^n [\int_B Q^{nm} \phi^m(x) dB] \tag{10}$$

Where

$$U^{nm} = \int_0^{t^+} \phi^s(x,t^-;y^i) \mu^m(\tau) d\tau \tag{11}$$

and

$$Q^{nm} = \int_0^{t^+} q^s(x,t^-;y^i) \eta^m(\tau) d\tau \tag{12}$$

Use of equations (9) and (11) gives the effect of the load at the field point x at time $t = t_n$. The effect in a categorized form can be found in (Dominguez, 1993; Sari, 2000). Also consideration of the integral (12) by using (8) gives the flux kernel explicitly in those works.

3. 1. Discretization

The boundary is discretized into a number of elements. Over each element, the co-ordinates are expressed by means of their nodal values by using linear elements whilst the field variables are represented by constant elements. The nodal values of the field variables on the boundary are approximated using the spatial interpolation function ψ_j for the node *j* so that (7) becomes:

$$\phi(x,\tau) = \sum_j \sum_m \eta^m(\tau)\psi_j(x)\phi^{mj}$$
$$q(x,\tau) = \sum_j \sum_m \mu^m(\tau)\psi_j(x)q^{mj} \tag{13}$$

Where ϕ^{mj} and q^{mj} denote the potential and its normal derivative at node j for time $t_m = m\Delta t$ whilst ψ_j is the spatial interpolation function for the field variables. When the boundary nodal quantities are constant over the element in approximation (13) $\psi_j = 1$. Using the spatial variations for the boundary node y^i with a set of discrete elements B_j, $j = 1$, 2,…, N on the boundary B, equation (10) can be written as:

$$\alpha^i \phi^{ni} = \sum_{m=1}^{n} \sum_{j=1}^{N} [\int_{B_j} U^{nm} \psi_j(x) dB] q^{mj}$$
$$- \sum_{m=1}^{n} \sum_{j=1}^{N} [\int_{B_j} Q^{nm} \psi_j(x) dB] \phi^{mj} \qquad (14)$$

In equation (14) n shows the final time, $t = n\Delta t$, whilst ϕ^{ni} denotes the unknown potential at the load point y^i, at time step n.

3. 1. 1. Regular Integrals

Here, the fundamental idea is to solve equation (4) numerically by discretizing boundary values spatially and temporally. The boundary B of the domain is discretized, as opposed to the domain techniques in which the domain is also discretized, to integrate spatially the kernels U^{nm} and Q^{nm} over the all boundary elements. The regular integrals are evaluated using a standard Gaussian quadrature. The integration to be evaluated is expressed by means of the homogeneous co-ordinate $-1 \leq \zeta \leq 1$ along the elements. To evaluate the integrals the differential is expressed, in the $x_1 x_2$-plane, as

$$dB = [(\frac{dx_1}{d\zeta})^2 + (\frac{dx_2}{d\zeta})^2]^{1/2} d\zeta = |J| d\zeta \qquad (15)$$

where $|J|$ is the Jacobian of the transformation. With the spatial discretization equation (14) takes the following form for the two-dimensional wave problems,

$$\alpha^i \phi^{ni} = \sum_{m=1}^{n} \sum_{j=1}^{N} [\int_{-1}^{1} U^{nm} \psi_j |J| d\zeta] q^{mj}$$
$$- \sum_{m=1}^{n} \sum_{j=1}^{N} [\int_{-1}^{1} Q^{nm} \psi_j |J| d\zeta] \phi^{mj} \qquad (16)$$

The boundary is divided into N elements and the field variables are assumed to be constant over each element and equal to the value at the mid-element node.

3. 1. 2. Improper Integrals

A singularity exists if and only if the load and field points coincide at the first time step. In the case of singular integrations, which arise when the field point is on the element being integrated, the integrals are treated analytically. In that case, the fundamental flux solution is zero since $\partial r / \partial n = 0$. The integral of the potential fundamental solution can be found in (Dominguez, 1993).

After evaluation of the regular and singular integrals, for each element j, one can write,

$$\sum_{m=1}^{n} \sum_{j=1}^{N} H^{nmij} \phi^{mj} - \sum_{m=1}^{n} \sum_{j=1}^{N} G^{nmij} q^{mj} = 0 \qquad (17)$$

Where,

$$G^{nmij} = \int_{B_j} U^{nm} \psi_j dB \qquad (18)$$

$$H^{nmij} = \begin{cases} \hat{H}^{nmij} + \alpha^i & \text{if } i = j \text{ and } n = m \\ \hat{H}^{nmij} & \text{if } i \neq j \text{ or } n \neq m \end{cases} \qquad (19)$$

With,

$$\hat{H}^{nmij} = \int_{B_j} Q^{nm} \psi_j dB \qquad (20)$$

Taking all boundary elements, equation (17) can be rewritten in a more abbreviated form:

$$\sum_{m=1}^{n} [H^{nm} \phi^m - G^{nm} q^m] = 0 \qquad (21)$$

Where G^{nm} and H^{nm} are square matrices which are calculated by spatial integration for each element and ϕ^m and q^m are the column vectors of boundary nodal quantities.

At time t, there are as many unknowns as the number of equations in the matrix equation (21). If the boundary quantities ϕ^m and q^m are known for the time m = 1, 2,…, n-1, then for each time step n, the solution can be found. The details of the solution procedure used here can be found in (Dominguez, 1993; Sarı, 2000).

4. NUMERICAL RESULTS

The examples presented below have been taken to demonstrate the usefulness of the boundary element implementation. To justify the BEM results, the FDM solutions of the problems were obtained. Also Reynolds (1978) used the FDM to obtain similar results. In the examples, the dynamical behaviour of the two-dimensional rock structures is examined. The boundary conditions are taken to be homogeneous and inhomogeneous for external and internal boundaries, respectively.

Consider the acoustic problem of solving equation (1) for a medium of finite and infinite extent. The geometries of the problem are shown in Figures 1 and 4, respectively. The wave speed for seawater is 1500 m/s. As can be seen from Figure 1, the receivers are accommodated in a horizontal line at 10 m below the top boundary.
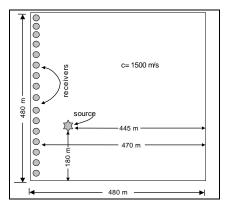

Figure 1. Physical the geometry of the medium used to generate seismograms

The seismograms correspond to the physical geometries of Figures 1 and 4 have time (in seconds) on the horizontal axis. All displacement values received at the selected receiver points are positioned on the vertical axis. The number of receivers used is 120. The distance between the adjacent receivers is equal. The vertical axis shows the length of the topside of the physical model in Figures 3 and 5, and its unit is meters (m). To obtain the synthetic seismograms the boundary elements used are uniform with 4 m element length and the total elapsed time is 1 s with 0.004 s the time increment. The source position is (180.445). In this work, the source is taken as an internal boundary, which is a small square. The Dirichlet boundary condition is prescribed for the sides of the small square with the length being 4 m. The Neumann boundary condition, $\partial \phi / \partial n = 0$, is specified for the topside. While the boundary quantities are defined using the Dirichlet boundary conditions for the other sides in Figure 2.

The results obtained from the BEM are in very good qualitative agreement on comparison to those found using the FDM to simulate a medium of infinite extent on the sides. Sari (2000) dealt with the quantitative differences between the BEM results and the FDM results. As expected there is no reflection from the sides or bottom, when the sides and the bottom at infinity.
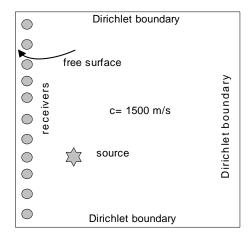

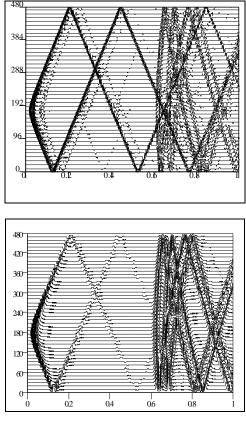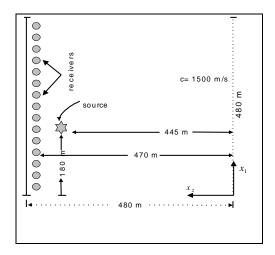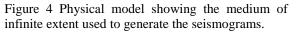Figure 2. Definition of the boundary conditions of the problem

Figure 3. Synthetic seismograms generated from the BEM and FDM solutions of the 2D acoustic wave equation for a medium of finite extent (from left to right, respectively)
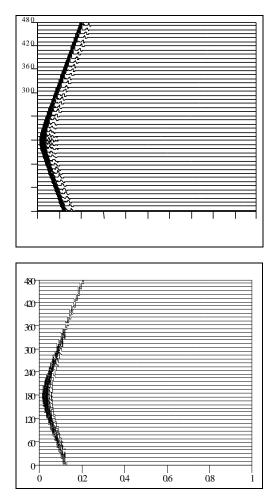
Figure 4 Physical model showing the medium of infinite extent used to generate the seismograms.

Figure 5 Synthetic seismograms generated from the BEM[(*)] and FDM solutions of the 2D acoustic wave equation for a medium of infinite extent (from left to right, respectively)

[(*)]: The BEM results in Figure 5 submitted in a paper to an International Journal

5. CONCLUSIONS AND FUTURE WORK

In this paper the time-domain 2D BEM has been used for solving 2D transient acoustic wave propagation problems in bounded and unbounded geophysical structures. The results have been compared with the FDM solutions. Since the structure of the BEM is profoundly intricate, it is believed that the BEM results presented here are important.

It is also concluded that to increase the numerical stability of the results for the time domain direct BEM, an increase in the number of elements is suggested. However it should not be forgotten that if the elements' size is taken to be very small then the desired stability might not be obtained. An alternative way to increase the stability of the solution is to use high order spatial variation of the field variables.

A possible improvement to this approach includes the coupling of the BEM with the FEM. This would retain the advantages of the BEM, but not require the fundamental solutions and evaluation of singular integrals.

6. REFERENCES

Ahmad, S. and Banerjee, P. K.1988. Time Domain Transient Elastodynamic Analysis of 3D Solids By BEM, *International Journal of Numerical Methods in Engineering,* Vol. 26, pp. 1709-1728.

Beskos, D. E. 1997. Boundary Element Methods in Dynamic Analysis: Part II (1986-1996), *Applied Mechanics Review*, Vol. 50, No. 3, pp. 149-197.

Birgisson, B. and Crouch, S. L. 1998. Elastodynamic Boundary Element Method For Piecewise Homogeneous Media, *International Journal for Numerical Methods in Engineering*, Vol. 42, pp. 1045-1069.

Bonnet, M. 1998. *Boundary Integral Equation Methods For Solids and Fluids*, John Wiley and Sons Ltd., New York.

Brebbia, C. A. and Dominguez, J. 1992. *Boundary Elements-an Introductory Course*, Second Edition, CMP, Southampton, Boston.

Carrer, J. A. M. and Mansur, W. J. 1994. Space Derivatives in the Time Domain BEM Analysis For

the Scalar Wave Equation, *Engineering Analysis With Boundary Elements*, Vol. 13, pp. 67-74.

Cheung, Y. K., Tham, L. G. and Lei, Z. X. 1993. Wave Propagation in Layered Media by Time Domain BEM, *Earthquake Engineering and Structural Dynamics*, Vol., 22, pp. 225-244.

Demir, I. 1999. *Seismic Wave Modelling Using Finite Difference Methods*, *Ph.D. Thesis*, University of Glamorgan, UK.

Dominguez, J. 1993. *Boundary Element in Dynamics*, CMP, Southampton.

Eringen, A. C. and Şuhubi, E. S. 1975. *Elastodynamics, Vol. II, Linear Theory*, Academic Press, New York.

Gallego, R. and Dominguez, J. 1996. Hypersingular BEM for Transient Elastodynamics, *International Journal For Numerical Methods in Engineering*, Vol. 39, pp. 1681-1705.

Israil, A. S. M. and Banerjee, P. K. 1990. Advanced Development of Time-Domain BEM for 2D Scalar Wave Propagation, *International Journal For Numerical Methods in Engineering,* Vol. 29, pp. 1003-1020.

Mansur, W. J. 1983. A Time-Stepping Technique to Solve Wave Propagation Problems Using the Boundary Element Method, *Ph. D. Thesis*, University of Southampton, UK.

Mansur, W. J., Carrer, J. A. M. and Siqueira, E. F.

N. 1998. Time Discontinuous Linear Traction Approximation in Time-Domain BEM Scalar Wave Propagation Analysis, *International Journal For Numerical Methods in Engineering*, Vol. 42, pp. 667-683.

Meijs, J. W. H, Weier, O. W., Peters, M. J. and Oosterom, A.V. 1989. On the Numerical Accuracy of the Boundary Element Method, *IEEE Transactions on Biomedical Engineering*, Vol. 36, No. 10, pp. 1038-1049.

Morse, P. M. and Feshbach, H. 1953. *Methods of Theoretical Physics*, McGraw-Hill.

Reynolds, A. C. 1978. Boundary Conditions For the Numerical Solution of Wave Propagation Problems, *Geophysics*, Vol. 43, No 6, pp. 1099-1110.

Richter, C. 1997. *Topics in Engineering*: *A Green's Function Time-Domain BEM of Elastodynamics, Brebbia, C. A. and Connor, J. J.* (*Eds*.), Vol. 31, CMP, Southampton, UK.

Sari, M. 2000. Seismic Wave Modelling Using the Boundary Element Method, *Ph.D. Thesis*, University of Glamorgan, UK, 2000.

Wang, C. and Takemiya, H. 1992. Analytical Elements of Time Domain BEM For Two-Dimensional Scalar Wave Problems, *International Journal for Numerical Methods in Engineering,* Vol. 33, pp. 1737-1754.

Zienkiewicz, O. C. 1977. *The finite Element Method*, Third Edition, London: McGraw-Hill.