THE TIME DOMAIN BOUNDARY ELEMENT METHOD FOR SCALAR WAVE PROBLEMS

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ABSTRACT

This paper deals with the stability of the boundary element method. The effects of various element sizes and time increments on the internal solution are analyzed. To this end, a time domain boundary element method is used. To achieve this, an existing BEM code for the boundary nodes is modified to optional internal nodes. Using appropriate time and spatial variations for the field variables, some observations on the numerical stability are reported. The internal solutions are presented for different β values and discussed the reasons of unstable cases appeared.

Key Words : Time domain BEM, Stability, Scalar wave

SKALAR DALGA PROBLEMLERİNİN ÇÖZÜMÜNDE ZAMAN-DOMENİ SINIR ELEMANI METODU

ÖZET

Bu makale Sınır Elemanı Metodunun (SEM) kararlılığını konu edinir. Farklı boyutlardaki elemanların ve değişik zaman artımlarının dahili çözüm üzerindeki etkileri analiz edilmiştir. Bunun yapılabilmesi için, bir zaman-domeni sınır elemanı metodu kullanılmıştır. Bu çalışma, sınır noktaları için mevcut olan bir SEM programının keyfi sayıdaki dahili noktalara modifiye edilerek başarıya ulaştırılmıştır. Alan değişkenlerinin nümerik hesaplanmasında benimsenen uygun zaman ve geometrik değişimler gözönünde tutularak nümerik kararlılık konusundaki bazı gözlemler kaleme alınmaktadır. Farklı β değerleri için dahili çözümler sunularak ve ortaya çıkan kararsız çözümlerin nedenleri irdelenmiştir.

Anahtar Kelimeler : Zaman domenli SEM, Kararlılık, Skalar dalga

1. INTRODUCTION

The number of works published on the time domain boundary element method (BEM) has been increased.

Mansur (1983) was the first to present the general formulation for the two-dimensional potential problems in time domain.

Antes (1985) generalized the formulation to more involved transient elastodynamic problems with arbitrary initial conditions.

In principle, the time or space interpolation functions are chosen arbitrarily. However when the piecewise linear time interpolation function is used for the flux, as well as for the potential, the solution process is prone to become unstable (Cole et al., 1978). In the present study the formulation based

upon the commonest (Sari, 2000) temporal variation is summarized first.

Growing evidence of numerical instabilities in the BEMs led some researchers to work on this problem. (Siebrits and Peirce, 1995 and Siebrits et al., 1997) studied the stability properties of the direct and indirect time domain elastodynamic BEM and drew attention to evidence of instabilities. Peirce and Siebrits (1996) used model problems to investigate the stability properties in the method. Peirce and Siebrits (1997) again and Birgisson et al. (1999) focused on the problem and suggested some methods to improve the numerical stability of the method. Arai et al. (1999) also joined the discussion with a paper using the Laplace transformation for two-dimensional elastodynamic problems. Yu et al. (1999) suggested using the linear temporal variation for traction as well, in terms of the so-called linear θ method, without any mathematical proof. Yu et al. (2000) used Galerkin type formulations to improve stability in the BEM scalar wave propagation analysis in an example. In most of the works carried out on the stability, the elastodynamic problems are used. In the present work, an acoustic problem is adopted to analyze the stability of the BEM.

In numerical analysis, the explicit finite different methods (FDM) and finite element methods (FEM) have the Courant-Fredericks-Lewy (CFL) condition on the time step. However the BEM has not got any such criteria and the CFL condition cannot be directly applied to the BEM schemes since they are based on a different discretization.

Therefore, an analysis of the sensitivity of the internal solutions to varying time steps is done for the scalar wave problems in the present paper. The effect of boundary discretizations is also discussed.

2. THE INTEGRAL EQUATION

The differential equation governing 2D wave equation for a homogeneous isotropic elastic body Ω enclosed by the boundary B can be written as (Morse and Feshbach, 1953):

$$c^2\phi_{,ii} + f = \ddot{\phi} \qquad (1)$$

In this equation ϕ, f and $\ddot{\phi}$ are functions of both position and time, and stand for potential displacement, body source and acceleration respectively, whilst c is the wave speed. Also in

equation (1), $\phi_{,ii}$ and $\ddot{\phi}$ are the second derivatives of the potential ϕ with respect to the direction x_i and the time t, respectively.

Taking the corresponding domain Ω with its boundary B; the fundamental solutions and the actual states of the differential equation (1) can be combined through the use of the dynamical reciprocity theorem (Banerjee, 1994), to give the following time-domain potential boundary integral equation,

$$\alpha\phi(y^i,t) = \int_0^{t^+}\!\!\int_B \phi^s(x,t^*;y^i)q(x,\tau)dBd\tau$$
$$\qquad\qquad - \int_0^{t^+}\!\!\int_B q^s(x,t^*;y^i)\phi(x,\tau)dBd\tau \qquad (2)$$

Where $t^* = t - \tau$. Equation (2) states the scalar potential field at any point of a medium as a function the scalar fields on the boundary. Here α^i depends only upon the local geometry of the body at the load point y^i. In equation (2), body source and initial conditions are neglected. Also in the equation:

$$\phi^s(x,t^*;y^i) = \frac{cH(ct^* - r)}{2\pi(c^2t^{*2} - r^2)^{1/2}} \qquad (3)$$

$$q^s = \frac{\partial\phi^s(x,t^*;y^i)}{\partial n} = \frac{\partial r}{\partial n}\frac{cH(ct^* - r)}{2\pi(c^2t^{*2} - r^2)^{3/2}} \qquad (4)$$

Where $r = \left|x - y^i\right|$ and H denotes the Heaviside function; ϕ and ϕ^s stand for the actual and fundamental solution states of the scalar potential respectively. Here the upper limit t^+ is used to avoid ending the integration at the peak of the Dirac delta function (Morse and Feshbach, 1953).

Since the radiation conditions (Eringen and Şuhubi, 1975) are automatically satisfied, the boundary integral equation (2) remains valid for unbounded media in addition to bounded media.

3. BEM FORMULATION

The boundary is discretized into a number of straight-line elements. Over each element, the co-ordinates are expressed by means of their nodal

values by using linear elements whilst the field variables are represented by constant elements.

In principle, the time or space interpolation functions are chosen arbitrarily. However when the piecewise linear time interpolation function is used for the flux, as well as for the potential, the solution process is prone to become unstable (Cole et al., 1978). Both for the potential and the flux, the piecewise constant time interpolation function was used for elastodynamic case by Spyrakos and Antes, (1986). However, Dominguez (1993) and Richter (1997) showed that in some cases, this approach gave poorer results than for the elements used in this study. Tian (1990) also used the constant time interpolation function for both field variables, and stated that this approach to be less stable than the linear time interpolation function used for both approximations.

Thus, in this study the predominant temporal variation (Sari, 2000) is used to obtain a numerical solution of the partial differential equation. In the abovementioned work, the potential and its normal derivative are interpolated by linear and constant variations in time, respectively.

Potentials and fluxes along the boundary are approximated using the interpolation functions,

$$\phi(x,\tau) = \sum_j \sum_m \eta^m(\tau)\psi_j(x)\phi^{mj}$$

$$q(x,\tau) = \sum_j \sum_m \mu^m(\tau)\psi_j(x)q^{mj} \qquad (5)$$

where ϕ^{mj} and q^{mj} stand for the potential and its normal derivative at node j for time $t_m = m\Delta t$ whilst ψ_j is the spatial interpolation function for the field variables. When the boundary nodal variables are constant over the element in approximation (5) $\psi_j = 1$. The temporal interpolation functions $\eta^m(\tau)$ and $\mu^m(\tau)$ explicitly are:

$$\eta^m(\tau) = \begin{cases} \dfrac{(\tau - \tau_{m-1})}{\Delta t} & \text{if } \tau_{m-1} < \tau < \tau_m \\[2mm] \dfrac{(\tau_{m+1} - \tau)}{\Delta t} & \text{if } \tau_m < \tau < \tau_{m+1} \\[2mm] 0 & \text{else} \end{cases} \qquad (6)$$

$$\mu^m(\tau) = \begin{cases} 1 & \text{if } \tau_{m-1} < \tau < \tau_m \\ 0 & \text{otherwise} \end{cases} \qquad (7)$$

The discretized form of equation (2) is:

$$\alpha^i \phi^{ni} = \sum_{m=1}^{n} \sum_{j=1}^{N} [\int_{B_j} U^{nm}\psi_j dB]q^{mj}$$
$$- \sum_{m=1}^{n} \sum_{j=1}^{N} [\int_{B_j} Q^{nm}\psi_j dB]\phi^{mj} \qquad (8)$$

Where;

$$U^{nm} = \int_0^{t^+} \phi^s(x,t^*;y^i)\mu^m(\tau)d\tau \qquad (9)$$

and

$$Q^{nm} = \int_0^{t^+} q^s(x,t^*;y^i)\eta^m(\tau)d\tau \qquad (10)$$

Categorized form of equations (9) and (10) can be found in Dominguez (1993) and Sari (2000). In equation (8), $t = n\Delta t$ and ϕ^{ni} denotes the unknown potential at the load point y^i, at the final time step.

3. 1. Evaluation of Integrals

Here, the main idea is to solve equation (4) numerically using spatial and temporal variations of boundary values. There are two types of integrals, singular and non-singular. The integrals become singular when $r \to 0$ for the first time step. In the case of singularity or $n = m$, the integrations along the boundary elements shown by equation (8) are treated analytically (Dominguez, 1993). The non-singular integrals are evaluated using a standard Gaussian quadrature with ten points. In the $x_1 x_2$-plane, to evaluate the integrals the differential is expressed as:

$$dB = [(\frac{dx_1}{d\zeta})^2 + (\frac{dx_2}{d\zeta})^2]^{1/2}d\zeta = |J|d\zeta \qquad (11)$$

Where $|J|$ is the Jacobian of the transformation. With the discretization, equation (8) takes the following form,

$$\alpha^i \phi^{ni} = \sum_{m=1}^{n} \sum_{j=1}^{N} [\int_{-1}^{1} U^{nm}\psi_j |J|d\zeta]q^{mj}$$
$$- \sum_{m=1}^{n} \sum_{j=1}^{N} [\int_{-1}^{1} Q^{nm}\psi_j |J|d\zeta]\phi^{mj} \qquad (12)$$

The boundary *B* is discretized into B_j, *j* = 1, 2,...,*N*, and the field variables are assumed to be equal to the value at the mid-element node for each element. After evaluation of all integrals, one can obtain the following linear algebraic equation system,

$$A^n X^n = R^n \qquad (13)$$

where the right hand side R^n is the sum of terms over the previous steps, and the known boundary conditions for time step *n* multiplied by their coefficient matrix. In the last equation, X^n and A^n are respectively the unknown vector and the system matrix for time step *n*. The details of the solution procedure used here can be found in (Dominguez, 1993).

4. APPLICATIONS

In order to analyse the effects of time step size and elements size on the stability of the method, a scalar wave problem was solved at one of the selected internal points (25.470) using the BEM formulation due to a source located in a geophysical structure.

If a solution process is divergent or begins to oscillate, then the process is called unstable. If a solution process requires a time step restriction from the user, then it is called conditionally stable, otherwise it is unconditionally stable. In general, a mathematical discussion on stability can be found in (Gilbert and Knobs, 1967).

The CFL condition provides an upper bound on the time step for the explicit FDM and the FEM in numerical analysis. However there are no such criteria in the BEM and the CFL condition cannot be directly applied to the BEM schemes since they are based on a different discretization and formulation of the system (Pierce Siebrist, 1996). Since the discretized BEM equations are very complicated, researchers, for example, Pierce Siebrist, (1996), consider model problems to determine the stability properties.

A convenient factor for measuring stability is $\beta = c\Delta t / \Delta x$, with time step size Δt, and the length of element Δx.

The material velocity for seawater is 1500 m/s. The physical geometry of the problem is given in Figure 1, and boundary conditions prescribed are shown in Figure 2. Homogeneous and inhomogeneous boundary conditions are used for the

external and internal (source) boundaries. In the model geometry, the source is located 35 m below the boundary while the receivers are placed in a horizontal line at 10 m below the top boundary.

The lengths $(\Delta x)_1$, $(\Delta x)_2$ and factors β^1, β^2 are for the external and internal boundaries, respectively. The physical system is a conservative system, so there is no energy loss.

Sensitivity of time step size and element size has been observed in the solutions in Figures 3. The effect of the uniformity of the elements to the solutions was analyzed considering β^1 and β^2.
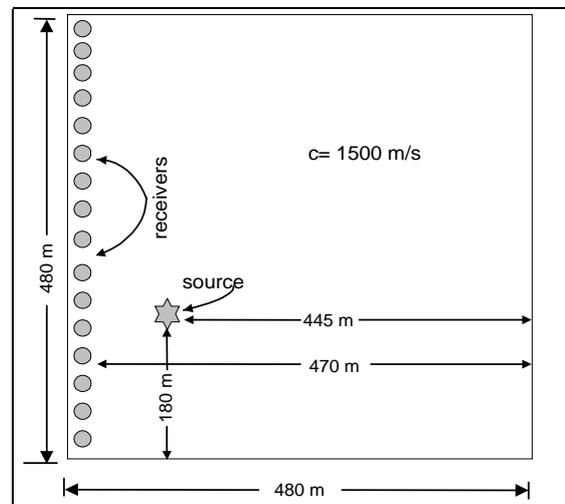

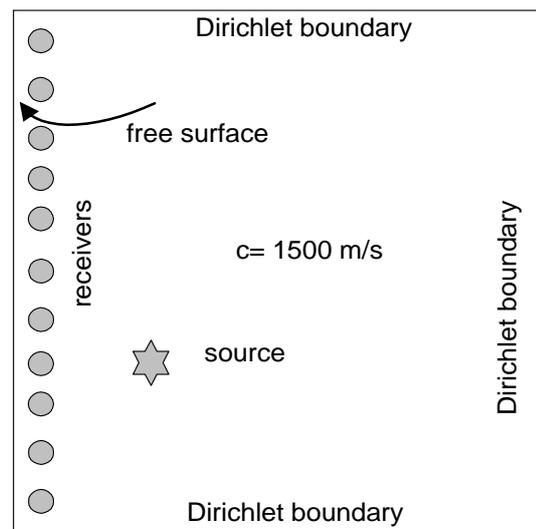Figure 1. Physical the geometry of the medium used to generate seismograms in the scalar BEM program

Figure 2. Definition of the boundary conditions of the problem

In Figures 3a to 3c, the element lengths are $(\Delta x)_1 = 5\,\text{m}$, $(\Delta x)_2 = 2\,\text{m}$. In these cases, the problem was solved for different β values. When the low β^1 and β^2 values have been adopted, the solution begins to oscillate (Figures 3a, 3d and 3h), and the resulting disturbances do not remain arbitrarily small throughout the wave path. To obtain a stable solution, following the application of arbitrary small perturbations the disturbances must remain arbitrarily small throughout the period of the investigation (Gilbert Knobs, 1967). When the waves reach an obstacle, they are partly reflected and partly transmitted (Graff, 1975).

When high β^1 and β^2 values are used (see Figures 3c, 3g and 3j), small responses in the medium cannot be seen. In other words, for very large time step size small disturbances in the body are invisible.
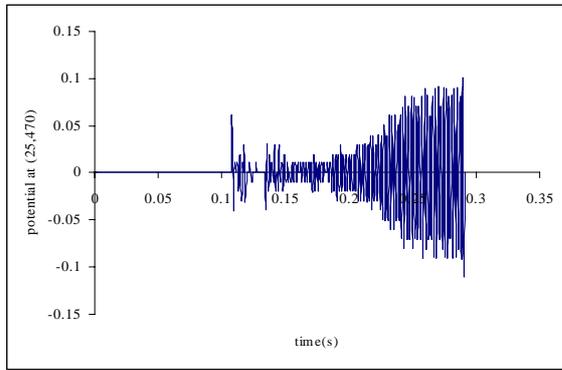

Figure 3a. Potential ϕ versus time; for point (25.470) with $\beta^1 = 0.15$, $\beta^2 = 0.375$, $(\Delta x)_1 = 5\,\text{m}$, $(\Delta x)_2 = 2\,\text{m}$
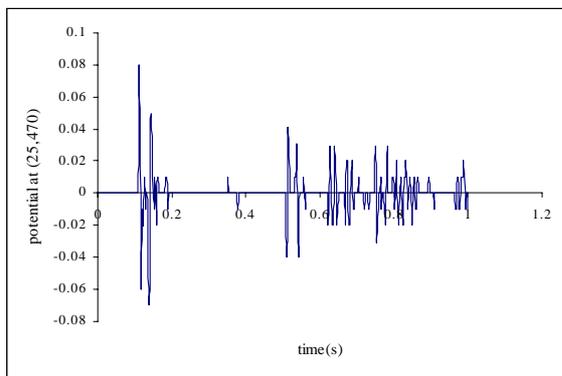

Figure 3b. Potential ϕ versus time; for point (25.470) with $\beta^1 = 0.6$, $\beta^2 = 1.5$, $(\Delta x)_1 = 5\,\text{m}$, $(\Delta x)_2 = 2\,\text{m}$
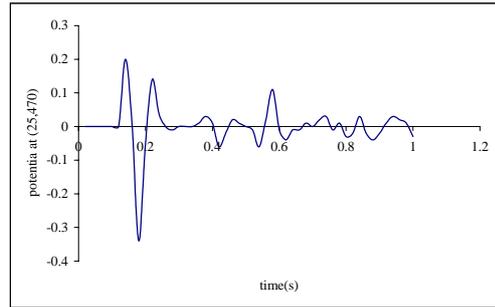

Figure 3c. Potential ϕ versus time; for point (25.470) with $\beta^1 = 6.$, $\beta^2 = 15.$, $(\Delta x)_1 = 5\,\text{m}$, $(\Delta x)_2 = 2\,\text{m}$
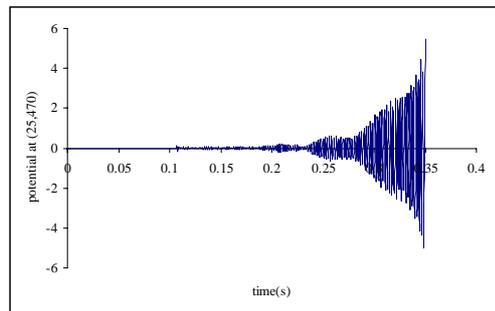

Figure 3d. Potential ϕ versus time; for point (25,470) with $\beta^1 = \beta^2 = 0.15$, $(\Delta x)_1 = (\Delta x)_2 = 5\,\text{m}$
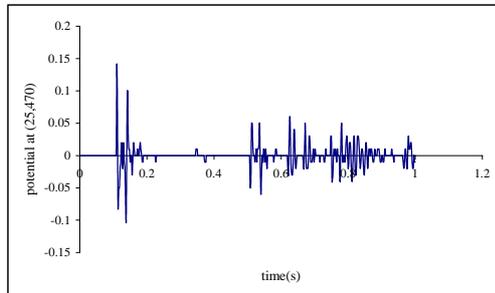

Figure 3e. Potential ϕ versus time; for point (25.470) with $\beta^1 = \beta^2 = 0.6$, $(\Delta x)_1 = (\Delta x)_2 = 5\,\text{m}$
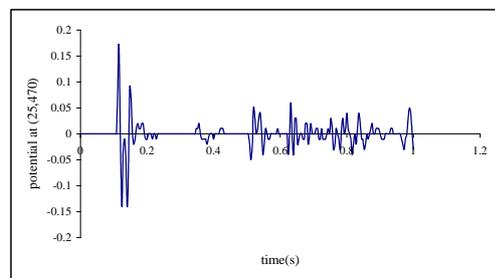

Figure 3f. Potential ϕ versus time; for point (25.470) with $\beta^1 = \beta^2 = 1.2$, $(\Delta x)_1 = (\Delta x)_2 = 5\,\text{m}$

The effect of the element size to the solution has also been observed. In Figures (3h), (3i) and (3j) the element length is larger than those of Figures (3a), (3b) and (3c), while the β^1 and β^2 values are equal. Comparison of Figures (3b) and (3i) suggests the larger element size represents the wave motion less accurately. Therefore, increasing the number of elements is suggested. However, it should not be forgotten that if the element size is taken to be very small, the desired stability may not be obtained. Because adoption of a small time step size means more time steps used and more accumulation of numerical errors. So this speeds up the instability of the solution.
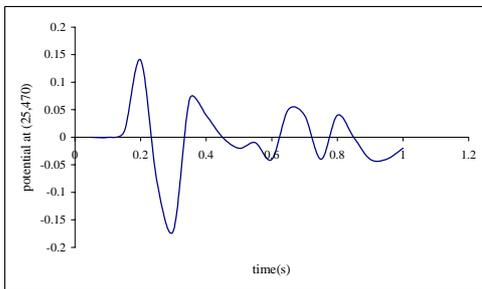

Figure 3g. Potential ϕ versus time; for point (25.470) with $\beta^1 = \beta^2 = 15.$, $(\Delta x)_1 = (\Delta x)_2 = 5$ m
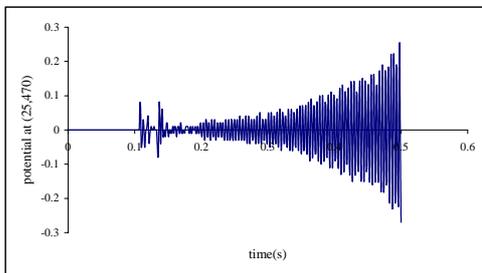

Figure 3h. Potential ϕ versus time; for point (25.470) with $\beta^1 = 0.15$, $\beta^2 = 0.375$, $(\Delta x)_1 = 10$ m, $(\Delta x)_2 = 4$ m
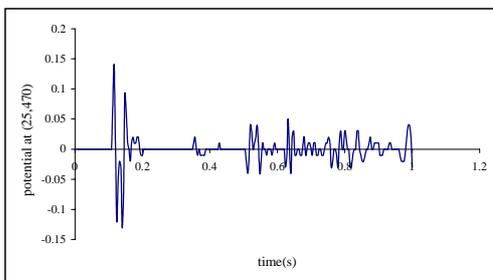

Figure 3i. Potential ϕ versus time; for point (25.470) with $\beta^1 = 0.6$, $\beta^2 = 1.5$, $(\Delta x)_1 = 10$ m, $(\Delta x)_2 = 4$ m
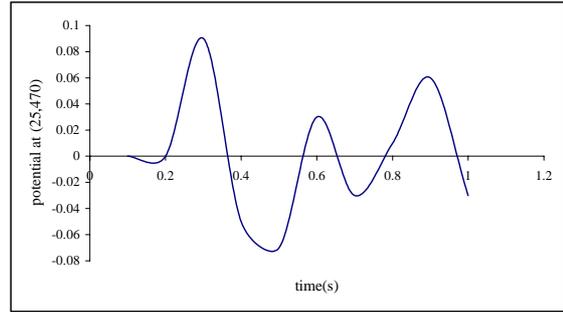

Figure 3j. Potential ϕ versus time; for point (25.470) with $\beta^1 = 15.$, $\beta^2 = 37.5$, $(\Delta x)_1 = 10$ m, $(\Delta x)_2 = 4$ m

Consideration of Figure (3b) tells us that even small reflections can be seen in the domain. Comparisons of Figures (3b-c), (3e-g) and (3i-j) lead us to see a solution with unseen reflections for the larger β^1 and β^2 values.

As can be seen from results (3b), (3f) and (3i), the reflections are weaker than the first wave. This is because the amplitude of the wave decreases as the radius increases since the energy in the wave front is spread out over an ever-increasing circumference.

In results (3e) and (3f), $(\Delta x)_1 = (\Delta x)_2$ and so that $\beta = \beta^1 = \beta^2$. In the first, β is 0.6 and in the second 1.2. In the first case, the solution looks to be more sensitive to selection of time step, while the second is not as clear as in Figure (3b). So for uniform elements β close to 1 is recommended. In the meantime, Figure (3b) suggests taking β close to 1 for non-uniform elements, referring to an intermediate element. In this respect, use of uniform elements can increase the stability of the numerical solution.

A similar analysis for boundary points was carried out in a different medium by Dominguez and Gallego, (1991) for elastodynamic problems, and their observations agree with our observations for internal points.

5. CONCLUSIONS AND RECOMMENDATIONS

A time domain direct BEM was employed for the solution of the problem.

It was observed that the solutions are sensitive for the selection of time step and element size. Perhaps one of the most important reasons for the sensitivity

is to use many equations during the numerical solution. Clearly, more equations give rise to more operations, which in turn cause a greater accumulation of round-off errors. Thus, the number of the linear equations affects the stability.

On the other hand, the desire for stable results requires more elements which in turn increase the dimension of the algebraic equation system. It must be emphasized that very few elements may not represent small changes as required. It is therefore recommended that a reasonable number of elements must be employed. This number can be determined by researchers depending on the nature of the problem to be solved.

Even though the example taken shows that the time domain-direct BEM is stable for practical applications, future work should concentrate on the analysis of numerical solutions, which give unconditional stability.

To determine the dynamical response of two-dimensional geophysical structures, the boundary has been composed of a number of 'linear lines'. The size of the source considered should not be too small to simulate meaningful changes in the media of interest.

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