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Some Estimates on Whitney Inequality for Differentiable Functions

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Abstract: In this study, we are interested in finding some estimates of the constants $W(k,r)$ $(k,r \in \mathbb{N})$, in the well-known Whitney Inequality for differentiable functions on the closed interval $[-1,1]$:

$$E_{k+r-1}\left(f,[-1,1]\right) \le W(k,r)\left(\frac{2}{k}\right)^{r} w_k\left(\frac{2}{k}, f^{(r)},[-1,1]\right).$$

Key Words: Whitney Inequality, divided differences, interpolation.

Diferansiyellenebilir Fonksiyonlar için Whitney Eşitsizliği Üzerine Bazı Sonuçlar

Özet: Bu çalışmada, $[-1,1]$ kapalı aralığı üzerinde diferansiyellenebilir fonksiyonlar için Whitney Eşitsizliği olarak bilinen:

$$E_{k+r-1}\left(f,[-1,1]\right) \le W(k,r)\left(\frac{2}{k}\right)^{r} w_k\left(\frac{2}{k}, f^{(r)},[-1,1]\right)$$

eşitsizliğindeki $W(k,r)$, $(k,r \in \mathbb{N})$, sabitleri için üst sınırların bulunması üzerinde durulmuştur.

Anahtar Kelimeler: Whitney Eşitsizliği, kesirli farklar, interpolasyon.

1. Introduction and Main Results

Let ¥ denote the set of natural numbers, ¥$_0 :=$ ¥ $\cup \{0\}$. We denote by $\mathbf{P}_n$, $n \in$ ¥$_0$, the space of algebraic polynomials of total degree at most n, by $C[a,b]$ the space of the real valued continuous functions on the closed interval $[a,b]$ equipped with the uniform norm:

$$\|f\|_{C[a,b]} := \max_{x \in [a,b]} |f(x)|$$

and by $C^r[a,b]$, $r \in$ ¥$_0$, the set all $r-$times continuously differentiable functions $f \in C[a,b]$; $C^0[a,b] := C[a,b]$. The deviation of $f \in C[a,b]$ from $\mathbf{P}_n$ is defined by

$$E_n(f,[a,b]) := \inf_{P_n \in \mathbf{P}_n} \|f - P_n\|_{C[a,b]}.$$

The purpose of the paper is to estimate the constants $W(k,r)$, $k,r \in$ ¥, in the well known Whitney Inequality: If $f \in C[a,b]$, $f \in C^r[a,b]$, then

$$E_{k+r-1}(f,[a,b]) \le W(k,r)\left(\frac{b-a}{k}\right)^r w_k\left(\frac{b-a}{k}, f^{(r)},[a,b]\right)$$

where

$$w_k(t,g,[a,b]) = \sup_{0<h\le t} \sup_{x \in [a,b-kh]} \left|\Delta_h^k g(x)\right|$$

is the $k-$th modulus of smoothness of the function g, and

$$\Delta_h^k g(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} g(x+jh)$$

is an $k-$th finite difference of g.

Many mathematicians have studied to estimate the Whitney constants: see, say, [1-8] for the references. Burkill [1] obtained the only known precise result: $W(2,0) = 1/2$. Whitney [2] proved that $1/2 \le W(k,0) < \infty$ for each $k \in$ ¥ and gave numerical estimates for $W(k,0)$ when $k \le 5$. In 1982, Sendov [3] conjectured that $W(k,0) \le 1$ for all k. However, this conjecture has been proved only for "small" k's: Whitney [2] for $k=3$, Kryakin [4] for $k=4$ and Zhelnov [5-6] for $k=5,6,7,8$. In general case, the most recent result is due to Gilewicz, Kryakin and Shevchuk [7] who proved that

$$W(k,0) \le 2 + 1/e^2 .$$

It follows from Lemma 3 in [8] that

$$W(k,1) \le 1/(e s_k), \qquad k \in \mathbb{N}$$

where $s_k = 1 + 1/2 + \ldots + 1/k$. For $r = 2,3,4$, the estimates of $W(k,r)$ were obtained in [9]:

$$W(k,r) \le \left(\frac{r}{e s_{k+r-1}} \right)^r, \qquad k \in \mathbb{N} .$$

Besides, in [9], the following estimates of $W(k,r)$ are obtained:

$$W(1,r) \le \frac{1}{r! 2^{2r+1} \cos \frac{\pi}{2(r+1)}}, \qquad r \in \mathbb{N},$$

$$W(2,r) \le \frac{1}{r! 2^{r^*} \cos^2 \frac{\pi}{2r^*}}, \qquad r \in \mathbb{N} ,$$

where $r^* := 2\lfloor (r+1)/2 \rfloor + 1$, where $\lfloor a \rfloor$ stands for the integral part of the number a.

The main results of the paper are the following.

Theorem 1. *For any* $f \in C^r[-1,1]$, *there is a polynomial* $P_{k+r-1} \in \mathbf{P}_{k+r-1}$ *such that*

$$\left| f(x) - P_{k+r-1}(x) \right| \le \frac{k^k}{2^k k! r!} \left(\left(1 - x^2\right)^{\lambda} \Pi(x) + \frac{3}{2^{k+r}} \right) \omega_k\left(2/k, f^{(r)}, [-1,1]\right).$$

$x \in [-1,1]$, *where* $\lambda := \lfloor (r+1)/2 \rfloor$ *and* $\Pi(x) = \prod_{j=0}^{k} (x + 1 - 2j/k)$.

Theorem 2. *We have*

$$W(k,r) \le \frac{1}{r!} \left(\frac{k(\lambda+1)}{e s_{k+\lambda}} \right)^{\lambda+1}$$

if $k > 2^{\lambda+1} + 1$.

In Section 2, will be given some relevant facts on divided differences, and in Section 3, we shall prove the Theorems 1 and 2.

2. Some Relevant Facts

In this section we shall give some auxiliary facts and notations which we will need in the proofs of the theorems.

Let $k\in \mathbb{N}$ and $\{y_j\}_{j=0}^{k}$ be a collection of distinct points $y_j\in[a,b]$. Recall, the divided difference of a function $g:[a,b]\to \mathbb{R}$ at the points $\{y_j\}_{j=0}^{k}$ is defined by

$$[y_0,y_1,\ldots,y_k;g]=\sum_{j=0}^{k}\frac{g(y_j)}{\prod_{i=0,i\neq j}^{k}\left(y_j-y_i\right)}.$$

By the definition, it can be easily seen that the equality

$$\int_c^d [y_0,y_1,\ldots,y_k;h(\cdot,y)]dy=[y_0,y_1,\ldots,y_k;\int_c^d h(\cdot,y)dy] \tag{2.1}$$

holds for any continuous function h defined on the rectangle $R:=[a,b]\times[c,d]$.

Denote by $L\left(x;g;y_0,y_1,\ldots,y_k\right)$ the Lagrange interpolation polynomial of degree $\leq k$ that interpolates the function g at the points y_j, $j=\overline{0,k}$. Then, as well known

$$g(x)-L\left(x;g;y_0,y_1,\ldots,y_k\right)=[x,y_0,y_1,\ldots,y_k;g]\prod_{j=0}^{k}\left(x-y_j\right).$$

Now, let $n\in \mathbb{N}$ and $\{x_i\}_{i=0}^{n}$ be a collection of points $x_i\in[a,b]$ that may coincide. Let $\{y_j\}_{j=0}^{k}$ be a collection of distinct points $y_j\in[a,b]$ such that each of $n+1$ points x_i coincides with one of the points y_j. Let a point y_j coincides exactly with s_j points x_i, then the number $p_j=s_j-1$ is called *multiplicity* of the point y_j. Clearly, $\sum_{j=0}^{k}s_j=n+1$, that is $\sum_{j=0}^{k}p_j=n-k$. Let a function $g\in C[a,b]$ have p_j first derivatives at a neighborhood of each point y_j. The generalized divided difference of order n of the function g at the points x_i, $i=0,1,...,n$, is defined by

$$[x_0,x_1,\ldots,x_n;g]:=\left(\prod_{j=0}^{k}\frac{1}{p_j!}\right)\frac{\partial^{n-k}}{\partial y_0^{p_0}\partial y_1^{p_1}\ldots\partial y_k^{p_k}}[y_0,y_1,...,y_k;g].$$

For $n=0$, set $[x_0;g]:=g\left(x_0\right)$. The generalized divided differences possess the same properties as the ordinary divided differences. Say, if $x_0\neq x_n$, then

$$[x_0,x_1,\ldots,x_n;g]=\frac{[x_1,x_2,\ldots,x_n;g]-[x_0,x_1,\ldots,x_{n-1};g]}{\left(x_n-x_0\right)}, \tag{2.2}$$

and let $L\left(x;g;x_0,x_1,\ldots,x_n\right)$ be the Hermite-Lagrange interpolation polynomial of degree $\leq n$, that interpolates the function g at the points $y_0,y_1,\ldots,y_k$ and interpolates all first p_j derivatives of g at the each point y_j, that is;

$$L^{(s)}\left(x;g;x_0,x_1,\ldots,x_n\right)=g^{(s)}(y_j),\quad j=\overline{0,k}, s=\overline{0,p_j}$$

where $g^{(0)}(x):=g(x)$, then

$$g(x)-L\left(x;g;x_0,x_1,...,x_k\right)=[x,x_0,x_1,\ldots,x_k;g]\prod_{j=0}^{k}\left(x-x_j\right). \tag{2.3}$$

The following lemma which proved in [9] enables to generalize the Lemma 3 of Zhuk and Natanson in [8].

Lemma 1. *Let* $r_0\in \mathbb{N}_0$, $n\in \mathbb{N}$, $r_0\le n$ *and* $\{x_i\}_{i=0}^{n}$ *be an arbitrary collection of points* $x_i\in[a,b]$. *If a function* $f\in C[a,b]$ *has the* r_0-1*-st absolutely continuous derivative on* $[a,b]$, *then*

$$[x_0,x_1,\ldots,x_n;f]=[x_r,x_{r+1},\ldots,x_n;f_r], \tag{2.4}$$

holds for each $r=0,1,\ldots,r_0$, *where* $f_0(x):=f(x)$, $f_1(x):=\int_0^1 f'(xt+(1-t)x_0)dt$ *and, for* $r>1$,

$$f_r(x):=\int_0^1\int_0^{t_1}\ldots\int_0^{t_{r-1}} f^{(r)}(xt_r+(t_{r-1}-t_r)x_{r-1}+\ldots+(1-t_1)x_0)dt_r\ldots dt_1.$$

3. Proofs of Theorems

Throughout this section, $[a,b]:=[-1,1]$, $\lambda:=\lfloor (r+1)/2\rfloor$ where $\lfloor a\rfloor$ stands for the integral part of a, and

$$x_j=\begin{cases}(-1)^s, & j=k+s,\ s=1,2,\ldots,2\lambda\\ -1+\dfrac{2j}{k}, & j=0,1,\ldots,k.\end{cases}$$

To shorten notation, we write $\|g\|$ and $\omega_k(g)$ instead of $\|g\|_{C[-1,1]}$ and $\omega_k(2/k,g,[-1,1])$, respectively.

Prof of Theorem 1. Let $L_{k+2\lambda}$ be the Hermite-Lagrange interpolation polynomial of degree $\le k+2\lambda$, which interpolates the function f at the points $x_0,x_1,\ldots,x_{k+2\lambda}$. By Newton's Formula, the coefficients of $x^{k+2\lambda}$ and $x^{k+2\lambda-1}$ in the polynomial $L_{k+2\lambda}$ are

$$A_{k+2\lambda}=[x_0,x_1,\ldots,x_{k+2\lambda};f],$$

(see, for instance [10, p.120]) and

$$
\begin{aligned}
A_{k+2\mathbf{l}-1} &= [x_0, x_1, \mathbf{L}, x_{k+2\mathbf{l}-1}; f] + A_{k+2\mathbf{l}} \\
&= \frac{[x_0, x_1, \mathbf{L}, x_{k+2\mathbf{l}-1}; f] + [x_0, x_1, \mathbf{L}, x_{k+2\mathbf{l}-2}, x_{k+2\mathbf{l}}; f]}{2},
\end{aligned}
$$

respectively.

Consider the polynomial

$$
P_{k+r-1}(x) = L_{k+2\mathbf{l}}(x) - \frac{A_{k+2\mathbf{l}}}{2^{k+2\mathbf{l}-1}} T_{k+2\mathbf{l}}(x) - (2\mathbf{l} - r)\frac{A_{k+2\mathbf{l}-1}}{2^{k+2\mathbf{l}-2}} T_{k+2\mathbf{l}-1}(x),
$$

of degree $\leq k+r-1$, where $T_n(x) = \cos\left(n \arccos x\right)$ is the n-th Chebyshev polynomial. The polynomial P_{k+r-1} is the desired one in Theorem 1. Let r be odd, i.e. $r = 2\mathbf{l} - 1$. Since $\|T_n\| = 1$, we conclude that

$$
\left| f(x) - P_{k+r-1}(x) \right| \leq \left| f(x) - L_{k+2\mathbf{l}}(x) \right| + \frac{\left|A_{k+2\mathbf{l}}\right|}{2^{k+2\mathbf{l}-1}} + \frac{\left|A_{k+2\mathbf{l}-1}\right|}{2^{k+2\mathbf{l}-2}} =: i_1 + i_2 + i_3.
$$

First we estimate $i_2 + i_3$. By using (2.2), (2.4) and (2.1), we obtain

$$
\begin{aligned}
A_{k+2\mathbf{l}} &= \frac{1}{2}\left([x_0, x_1, \mathbf{L}, x_{k+2\mathbf{l}-2}, x_{k+2\mathbf{l}}; f] - [x_0, x_1, \mathbf{L}, x_{k+2\mathbf{l}-1}; f]\right) \\
&= \frac{1}{2}\int_0^1 \int_0^{t_1} \mathbf{L} \int_0^{t_{r-1}} ([x_0, x_1, \mathbf{L}, x_k; g_0 - g_1] dt_r \, \mathbf{L} \, dt_1,
\end{aligned}
$$

and similar arguments provide

$$
A_{k+2\mathbf{l}-1} = \frac{1}{2}\int_0^1 \int_0^{t_1} \mathbf{L} \int_0^{t_{r-1}} [x_0, x_1, \mathbf{L}, x_k; g_0 + g_1] dt_r \, \mathbf{L} \, dt_1
$$

where $g_i(x) := f^{(r)}\left((1+x)t_r - 2\sum_{j=2}^{r-1}(-1)^j t_j + t_1 + (1-t_1)(-1)^i \right)$, $i = 0,1$. Since, for both $i = 0,1$,

$$
\begin{aligned}
\left\| [x_0, x_1, \mathbf{L}, x_k; g_i] \right\| &= \left| \frac{k^k}{2^k k!} \Delta^k_{2/k} g_i\left(x_0\right) \right| \leq \frac{k^k}{2^k k!} w_k\left(g_i\right) \\
&= \frac{k^k}{2^k k!} w_k\left(\frac{2}{k}t_r, f^{(r)}, [-1,1]\right) \leq \frac{k^k}{2^k k!} w_k\left(f^{(r)}\right)
\end{aligned}
$$

Then

$$
\begin{aligned}
i_2 + i_3 &= \frac{\left|A_{k+2\mathbf{l}}\right|}{2^{k+2\mathbf{l}-1}} + \frac{\left|A_{k+2\mathbf{l}-1}\right|}{2^{k+2\mathbf{l}-2}} \\
&\leq \frac{3k^k}{2^{2k+r} k!} \int_0^1 \int_0^{t_1} \mathbf{L} \int_0^{t_{r-1}} w_k\left(f^{(r)}\right) dt_r \, \mathbf{L} \, dt_1 = \frac{3k^k}{2^{2k+r} k! r!} w_k\left(f^{(r)}\right).
\end{aligned}
$$

Let us now estimate i_1. By using (2.2), (2.3), (2.4) and (2.1), we obtain

$$f(x)-L_{k+2\mathbf{l}}(x)=\left(x^2-1\right)^{\mathbf{l}}\Pi(x)\left[x_0,x_1,\mathbf{L},x_{k+2\mathbf{l}},x;f\right]$$

$$=\frac{\left(x^2-1\right)^{\mathbf{l}}\Pi(x)}{4}\Big(\left[x_0,x_1,...,x_{k+2\mathbf{l}-4},x_{k+2\mathbf{l}-2},x_{k+2\mathbf{l}},x;f\right]$$

$$-2\left[x_0,x_1,...,x_{k+2\mathbf{l}-2},x;f\right]+\left[x_0,x_1,...,x_{k+2\mathbf{l}-3},x_{k+2\mathbf{l}-1},x;f\right]\Big)$$

$$=\frac{\left(x^2-1\right)^{\mathbf{l}}\Pi(x)}{4}\int_0^1\int_0^{t_1}\mathbf{L}\int_0^{t_{r-1}}[x_0,x_1,...,x_k;g_2-2g_3+g_4]dt_r\mathbf{L}\,dt_1.$$

where

$$g_i(u)=f^{(r)}\left(ut_r-2(t_{r-1}-t_{r-2}+\mathbf{L}+t_4)+(1-x)t_3+a_i\right)$$

$i=2,3,4$, where $a_2=1-(1-x)t_2$, $a_3=1-2t_1+(1+x)t_2$ and $a_4=(1+x)t_2-1$. Therefore, as in the estimation of i_2+i_3, we obtain

$$i_1=\left|f(x)-L_{k+2\mathbf{l}}(x)\right|\le\frac{k^k\left(1-x^2\right)^{\mathbf{l}}\left|\Pi(x)\right|}{2^k k!r!}w_k\left(f^{(r)}\right),$$

which completes the proof for the case r is odd, but the same conclusion can be drawn for the case r is even, in this manner, Theorem 1 is proved.

The following lemma will be needed in the proof Theorem 2. Set $h=2/k$, and recall, the logarithm with base $a>0(\neq 1)$, is defined by $\log_a x:=\log x/\log a$, $x>0$.

Lemma 2. *Let* $k\ge 2$. *For* $\mathbf{l}+1<\log_2(k-1)$, *the equality*

$$\max_{x\in[-1,1]}\left|\left(x^2-1\right)^{\mathbf{l}}\Pi(x)\right|=\max_{x\in[-1,-1+2h]}\left|\left(x^2-1\right)^{\mathbf{l}}\Pi(x)\right|$$

holds.

Proof. For $-1+h\le y\le -h/2$, consider the function

$$H(y):=\left|\frac{\left((y+h)^2-1\right)^{\mathbf{l}}\Pi(y+h)}{\left(y^2-1\right)^{\mathbf{l}}\Pi(y)}\right|=\frac{(1+y+h)}{(1-y)}\left(1-\frac{h(2y+h)}{\left(1-y^2\right)}\right)^{\mathbf{l}}.$$

Since $H(-h/2)=1$, $H'(-h/2)>0$ and H' has only one zero in $[-1+h,-h/2]$ for $\mathbf{l}\le(k-2)(k+1)/(2k)$, it is sufficient to show that $H(-1+h)\le 1$. Indeed,

$$H(-1+h)=\frac{2}{k-1}\left(\frac{2(k-2)}{k-1}\right)^{\mathbf{l}}\le 1,$$

for $\mathbf{l}+1<\log_2(k-1)$. The proof is complete, since $\log_2(k-1)-1<(k-2)(k+1)/(2k)$, for all $k\geq 2$.

Proof of Theorem 2. In order to prove Theorem 2 it is enough to check the inequality

$$\frac{k^{k+r}}{2^{k+r}k!r!}\left(\left(1-x^2\right)^{\mathbf{l}}\left|\Pi(x)\right|+\frac{3}{2^{k+r}}\right)\leq\frac{1}{r!}\left(\frac{k(\mathbf{l}+1)}{es_{k+\mathbf{l}}}\right)^{\mathbf{l}+1}, \tag{3.1}$$

$x\in\left[-1,1\right]$. First, we prove the estimate

$$\frac{\left|\left(x^2-1\right)^{\mathbf{l}}\Pi(x)\right|}{k^{-k}2^k k!}\leq\max\left\{\frac{1}{2}\left(\frac{4(\mathbf{l}+1)}{ek\left(s_k+\mathbf{l}/k\right)}\right)^{\mathbf{l}+1},\frac{1}{e^2}\left(\frac{4(\mathbf{l}+1)}{ek\left(s_k-2+\mathbf{l}/k\right)}\right)^{\mathbf{l}+1}\right\}, \tag{3.2}$$

for all $x\in[-1,1]$ and $\mathbf{l}+1<\log_2(k-1)$. Let $C_{k,\mathbf{l}}$ denote the right hand of (3.2).

If $-1<x<-1+h$ and $u=k(x+1)/2$ then $0<u<1$ and

$$\begin{aligned}\frac{\left|\left(x^2-1\right)^{\mathbf{l}}\Pi(x)\right|}{k^{-k}2^k k!}&=\frac{2^{2\mathbf{l}+1}}{k^{\mathbf{l}+1}}u^{\mathbf{l}+1}\left(1-\frac{u}{1}\right)\left(1-\frac{u}{2}\right)\mathbf{L}\left(1-\frac{u}{k-1}\right)\left(1-\frac{u}{k}\right)^{\mathbf{l}+1}\\&\leq\frac{2^{2\mathbf{l}+1}}{k^{\mathbf{l}+1}}u^{\mathbf{l}+1}\left(1-\frac{u\left(s_k+\mathbf{l}/k\right)}{k+\mathbf{l}}\right)^{k+\mathbf{l}}\\&\leq\frac{2^{2\mathbf{l}+1}u^{\mathbf{l}+1}e^{-\left(s_k+\mathbf{l}/k\right)u}}{k^{\mathbf{l}+1}}\leq\frac{1}{2}\left(\frac{4(\mathbf{l}+1)}{ek\left(s_k+\mathbf{l}/k\right)}\right)^{\mathbf{l}+1}.\end{aligned}$$

On the other hand, applying similar arguments to the case $-1+h<x<-1+2h$, and using Lemma 2, we obtain (3.2).

Now, taking into account (3.2) and the inequality $k!\geq k^k e^{-k}\sqrt{2pk}$ which follows from Stirling's formula, we get

$$r!W(k,r)\leq\left(\frac{k}{2}\right)^r C_{k,\mathbf{l}}+\frac{3e^k k^r}{4^{k+r}\sqrt{2pk}}.$$

It is easy to check that

$$\left(\frac{k}{2}\right)^r C_{k,\mathbf{l}}+\frac{3e^k k^r}{4^{k+r}\sqrt{2pk}}\leq\left(\frac{k(\mathbf{l}+1)}{es_{k+\mathbf{l}}}\right)^{\mathbf{l}+1},$$

for $\mathbf{l}+1<\log_2(k-1)$. Thus, Theorem 2 is proved.

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