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Asymptotic stability of ground states of quadratic nonlinear Schrödinger equation with potential in 4D

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Abstract. In this paper a class of quadratic nonlinear Schrödinger equation in four space dimensions with an attractive potential is considered. We investigate asymptotic stability of the nonlinear bound states, i.e. periodic in time localized in space solutions. We show that all solutions with small initial data, converge to a nonlinear bound state. Therefore, the non-linear bound states are asymptotically stable.

1. Introduction

In this paper we study the long time behavior of solutions of the nonlinear Schrödinger equation (NLS) with potential in four space dimensions (4-d):

$$i\partial_t u(t,x) = [-\Delta_x + V(x)]u + g(u), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^4 \tag{1}$$

$$u(0,x) = u_0(x) \tag{2}$$

where $g(u) = |u|u$ is quadratic nonlinearity. This nonlinear equation admits periodic in time, localized in space solutions (bound states or solitary waves). They can be obtained via both variational techniques [1, 18, 23] and bifurcation methods [13, 17, 18]. Moreover the set of periodic solutions can be organized as a manifold (center manifold). Orbital stability of solitary waves, i.e. stability modulo the group of symmetries $u \mapsto e^{-i\theta}u$, was first proved in [18, 25], see also [8, 9, 20].

Asymptotic stability studies of solitary waves were initiated in the work of A. Soffer and M. I. Weinstein [21, 22], see also [2, 3, 4, 6, 10]. Center manifold analysis was introduced in [17], see also [24]. In this (4d-5d makalesi) it was shown that solutions of (1)-(2) with small initial data converge to the orbit of a certain bound state. In this work it will be shown that convergence is asymptotic. The main challenge is to obtain good estimates for the semigroup of operators generated by the time dependent linearization that we use. This is accomplished in [11]. The technique is perturbative, and similar to the one developed by E. Kirr, A. Zarnescu and Ö. Mızrak for Schrödinger type operators in [11, 12, 13, 14]. The main difference between [11] is the decomposition of the dynamics. The decomposition used in [11] works for all nonlinearities. The decomposition used in this paper works for at least quadratic nonlinearities but gives asymptotic stability.

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Notations: $H = -\Delta + V$;

$L^p = \{f : \mathbb{R}^4 \mapsto \mathbb{C} \mid f$ measurable and $\int_{\mathbb{R}^4} |f(x)|^p dx < \infty\}$, $\|f\|_p = \left(\int_{\mathbb{R}^4} |f(x)|^p dx\right)^{1/p}$ denotes the standard norm in these spaces;

$< x >= (1 + |x|^2)^{1/2}$, and for $\sigma \in \mathbb{R}$, L^2_σ denotes the L^2 space with weight $< x >^{2\sigma}$, i.e. the space of functions $f(x)$ such that $< x >^\sigma f(x)$ are square integrable endowed with the norm $\|f(x)\|_{L^2_\sigma} = \| < x >^\sigma f(x)\|_2$;

$\langle f, g\rangle = \int_{\mathbb{R}^4} \overline{f}(x) g(x) dx$ is the scalar product in L^2 where $\overline{z}$ = the complex conjugate of the complex number z;

P_c is the projection on the continuous spectrum of H in L^2;

H^n denote the Sobolev spaces of measurable functions having all distributional partial derivatives up to order n in L^2, $\| \cdot \|_{H^n}$ denotes the standard norm in this spaces.

2. Preliminaries. The center manifold.

The center manifold is formed by the collection of periodic solutions for (1):

$$u_E(t, x) = e^{-iEt}\psi_E(x) \tag{3}$$

where $E \in \mathbb{R}$ and $0 \not\equiv \psi_E \in H^2(\mathbb{R}^4)$ satisfy the time independent equation:

$$[-\Delta + V]\psi_E + g(\psi_E) = E\psi_E \tag{4}$$

Clearly the function constantly equal to zero is a solution of (4) but (iii) in the following hypotheses on the potential V allows for a bifurcation with a nontrivial, one parameter family of solutions:

(H1) Assume that

(i) $V(x)$ sutisfies the following properties:

1. $< x >^\rho V(x) : H^\eta \to H^\eta$, for some $\rho > 8$ and $\eta > 0$;
2. $\nabla V \in L^p(\mathbb{R}^4)$ for some $2 \leqslant p \leqslant \infty$ and $|\nabla V(x)| \to 0$ as $|x| \to \infty$;
3. the Fourier transform of V is in L^1.

(ii) 0 is a regular point‡ of the spectrum of the linear operator $H = -\Delta + V$ acting on L^2.

(iii) H acting on L^2 has exactly one negative eigenvalue $E_0 < 0$ with corresponding normalized eigenvector ψ_0. It is well known that $\psi_0(x)$ is exponentially decaying as $|x| \to \infty$, and can be chosen strictly positive.

Conditions (i) and (ii) guarantee the applicability of dispersive estimates in [15] and **[?]** to the Schrödinger group $e^{-iHt}P_c$. Condition (i)2. implies certain regularity of the nonlinear bound states while (i)3. allows us to use commutator type inequalities,

‡see [19, Definition 7] or $M_\mu = \{0\}$ in relation (3.1) in [15]

see [12, Theorem 5.2]. All these are needed to obtain estimates for the semigroup of operators generated by our time dependent linearization, see Theorem 4.1 and 4.2 in [11]. In particular (i)1. implies the local well posedness in H^1 of the initial value problem (1)-(2), see section 3.

By the standard bifurcation argument in Banach spaces [16] for (4) at $E = E_0$, condition (iii) guarantees existence of nontrivial solutions. Moreover, these solutions can be organized as a C^1 manifold (center manifold), see [13, section 2]. Since our main result requires, we are going to show in what follows that the center manifold is C^2.

3. Main Result

Theorem 3.1. *Assume that hypothesis (H1) and either (H2) or (H2') hold. Then there exists an ε_0 such that for all initial conditions $u_0(x)$ satisfying*

$$\max\{\|u_0\|_{L^{p'}}, \|u_0\|_{H^1}\} \leq \varepsilon_0, \qquad \frac{1}{p'} + \frac{1}{p} = 1$$

the initial value problem (1)-(2) is globally well-posed in H^1 and the solution decomposes into a radiative part and a part that asymptotically converges to a ground state.

More precisely, there exist a C^1 function $a : \mathbb{R} \mapsto \mathbb{C}$ such that, for all $t \in \mathbb{R}$ we have:

$$u(t,x) = \underbrace{a(t)\psi_0(x) + h(a(t))}_{\psi_E(t)} + \eta(t,x) \tag{5}$$

where $\psi_E(t)$ is on the central manifold (i.e it is a ground state) and $\eta(t,x) \in \mathcal{H}_{a(t)}$, see [12]. Moreover there exists the ground states states $\psi_{E_{\pm\infty}}$ and the C^1 function $\theta : \mathbb{R} \mapsto \mathbb{R}$ such that $\lim_{|t|\to\infty} \theta(t) = 0$ and:

$$\lim_{t\to\pm\infty} \|\psi_E(t) - e^{-it(E_\pm - \theta(t))}\psi_{E_{\pm\infty}}\|_{H^2 \bigcap L^2_\sigma} = 0,$$

while η satisfies the following decay estimates:

$$\|\eta(t)\|_{L^p} \leq C\frac{\varepsilon_0}{(1+|t|)^{4(\frac{1}{2}-\frac{1}{p})}}, \quad 2 \leq p \leq \infty$$

where the constant C independent of ε_0.

Remark 3.1. *Our results for these cases are stronger than the ones in [17, 21, 22] because we do not require the initial condition to be in L^2_σ, $\sigma > 1$. Compared to [10] we have sharper estimates for the asymptotic decay to the ground state but we require the initial data to be in $L^{p'}$.*

Proof of Theorem 3.1 It is well known that under hypothesis (H1)(i) the initial value problem (1)-(2) is locally well posed in the energy space H^1 and its L^2 norm is conserved, see for example [5, Cor. 4.3.3 at p. 92]. Global well posedness follows via energy estimates from $\|u_0\|_{H^1}$ small, see [5, Remark 6.1.3 at p. 165].

We choose $\varepsilon_0 \leqslant \delta_1$ given by Lemma 2.1 in [12]. Then, for all times, $\|u(t)\|_{L^2} \leqslant \delta_1$ and we can decompose the solution into a solitary wave and a dispersive component as in (5):

$$u(t) = a(t)\psi_0 + h(a(t)) + \eta(t) = \psi_E(t) + \eta(t)$$

Moreover, by possible making ε_0 smaller we can insure that that $\|u(t)\|_{L^2} \leqslant \varepsilon_0$ implies $|a(t)| \leqslant \delta_2,\ t \in \mathbb{R}$ where δ_2 is given by Lemma 2.2 in [12]. In addition, since

$$u \in C(\mathbb{R}, H^1(\mathbb{R}^4)) \cap C^1(\mathbb{R}, H^{-1}(\mathbb{R}^4)),$$

and $u \mapsto a$ respectively $u \mapsto \eta$ are C^1, we get that $a(t)$ is C^1 and $\eta \in C(\mathbb{R}, H^1) \cap C^1(\mathbb{R}, H^{-1})$.

The solution is now described by the C^1 function $a : \mathbb{R} \in \mathbb{C}$ and $\eta(t) \in C(\mathbb{R}, H^1) \cap C^1(\mathbb{R}, H^{-1})$. To obtain estimates for them it is useful to remove their dominant phase. Consider the C^2 function:

$$\theta(t) = \int_0^t E(|a(s)|)ds$$

and

$$\tilde{u}(t) = e^{i\theta(t)}u(t),$$

then $\tilde{u}(t)$ satisfies the differential equation:

$$i\partial_t \tilde{u}(t) = -E(|a(t)|)\tilde{u}(t) + (-\Delta + V)\tilde{u} + |\tilde{u}(t)|\tilde{u}(t), \tag{6}$$

moreover, like $u(t)$, $\tilde{u}(t)$ can be decomposed:

$$\tilde{u}(t) = \underbrace{\tilde{a}(t)\psi_0 + h(\tilde{a}(t))}_{\tilde{\psi}_E(t)} + \tilde{\eta}(t) \tag{7}$$

where

$$\tilde{a}(t) = e^{i\theta(t)}a(t), \quad \tilde{\eta}(t) = e^{i\theta(t)}\eta(t) \in \mathcal{H}_{\tilde{a}(t)}$$

By plugging in (7) into (6) we get

$$\begin{aligned} i\frac{\partial\tilde{\eta}}{\partial t} + iD\tilde{\psi}_E|_{\tilde{a}}\frac{d\tilde{a}}{dt} &= (-\Delta + V - E)(\tilde{\psi}_E + \tilde{\eta}) + g(\tilde{\psi}_E) + g(\tilde{\psi}_E + \tilde{\eta}) - g(\tilde{\psi}_E) \\ &= L_{\tilde{\psi}_E}\tilde{\eta} + F_2(\tilde{\psi}_E, \tilde{\eta}) \end{aligned}$$

or, equivalently,

$$\frac{\partial\tilde{\eta}}{\partial t} + \underbrace{\frac{\partial\tilde{\psi}_E}{\partial a_1}\frac{da_1}{dt} + \frac{\partial\tilde{\psi}_E}{\partial a_2}\frac{da_2}{dt}}_{\in \mathrm{span}_{\mathbb{R}}\{\frac{\partial\tilde{\psi}_E}{\partial a_1}, \frac{\partial\tilde{\psi}_E}{\partial a_2}\}} = \underbrace{-iL_{\tilde{\psi}_E}\tilde{\eta}}_{\in\mathcal{H}_{\tilde{a}}} - iF_2(\tilde{\psi}_E, \tilde{\eta}) \tag{8}$$

where $L_{\tilde{\psi}_E}$ is defined by

$$L_{\tilde{\psi}_E}\tilde{\eta} = (-\Delta + V - E)\tilde{\eta} + \frac{d}{d\varepsilon}g(\tilde{\psi}_E + \varepsilon\tilde{\eta})|_{\varepsilon=0}$$

and F_2 denotes the nonlinear terms in $\tilde\eta$

$$F_2(\tilde\psi_E,\tilde\eta) = g(\tilde\psi_E+\tilde\eta) - g(\tilde\psi_E) - \frac{d}{d\varepsilon} g(\tilde\psi_E+\varepsilon\tilde\eta)|_{\varepsilon=0} \tag{9}$$

and we also used the fact that $\tilde\psi_E$ is a solution of the eigenvalue problem (4).

We now project (8) onto the invariant subspaces of $-iL_{\tilde\psi_E}$, namely span$\{\frac{\partial\tilde\psi_E}{\partial a_1}, \frac{\partial\tilde\psi_E}{\partial a_2}\}$, and $\mathcal{H}_{\tilde a}$.

$$\begin{bmatrix} \Re\langle\Psi_1(\tilde a), \frac{\partial\tilde\eta}{\partial t}\rangle \\ \Re\langle\Psi_2(\tilde a), \frac{\partial\tilde\eta}{\partial t}\rangle \end{bmatrix} + \frac{d}{dt}\begin{bmatrix} \tilde a_1 \\ \tilde a_2 \end{bmatrix} = \begin{bmatrix} F_{21}(\tilde\psi_E,\tilde\eta) \\ F_{22}(\tilde\psi_E,\tilde\eta) \end{bmatrix}$$

where $\Psi_{1,2}$ are given by

$$\Psi_1(a_1,a_2) = -i\frac{\partial\psi_E}{\partial a_2}\left(\Re\langle -i\frac{\partial\psi_E}{\partial a_2},\ \frac{\partial\psi_E}{\partial a_1}\rangle\right)^{-1}, \qquad \Psi_2(a_1,a_2) = i\frac{\partial\psi_E}{\partial a_1}\left(\Re\langle i\frac{\partial\psi_E}{\partial a_1},\ \frac{\partial\psi_E}{\partial a_2}\rangle\right)^{-1}. \tag{10}$$

$$F_{2j} = \Re\langle\Psi_j, -iF_2(\tilde\psi_E,\tilde\eta)\rangle, \quad j=1,2. \tag{11}$$

To calculate $\Re\langle\Psi_j, \frac{\partial\tilde\eta}{\partial t}\rangle$, $j=1,2$ we use the fact that $\tilde\eta\in\mathcal{H}_{\tilde a}$, for all $t\in\mathbb{R}$, i.e.

$$\Re\langle\Psi_j(\tilde a(t)), \tilde\eta(t)\rangle \equiv 0$$

Differentiating the latter with respect to t we get:

$$\Re\langle\Psi_j, \frac{\partial\tilde\eta}{\partial t}\rangle = -\Re\langle\frac{\partial\Psi_j}{\partial a_1}\frac{d\tilde a_1}{dt} + \frac{\partial\Psi_j}{\partial a_2}\frac{d\tilde a_2}{dt}, \tilde\eta\rangle, \quad j=1,2$$

which replaced above leads to:

$$\frac{d}{dt}\begin{bmatrix} \tilde a_1 \\ \tilde a_2 \end{bmatrix} = (\mathbb{I}_{\mathbb{R}^2} - M_{\tilde u})^{-1}\begin{bmatrix} F_{21}(\tilde\psi_E,\tilde\eta) \\ F_{22}(\tilde\psi_E,\tilde\eta) \end{bmatrix} \tag{12}$$

where the two by two matrix $M_{\tilde u}$ is the Jacobi matrix given in [12]. In particular

$$\begin{bmatrix} \Re\langle\Psi_1, \frac{\partial\tilde\eta}{\partial t}\rangle \\ \Re\langle\Psi_2, \frac{\partial\tilde\eta}{\partial t}\rangle \end{bmatrix} = -M_{\tilde u}(\mathbb{I}_{\mathbb{R}^2} - M_{\tilde u})^{-1}\begin{bmatrix} F_{21}(\tilde\psi_E,\tilde\eta) \\ F_{22}(\tilde\psi_E,\tilde\eta) \end{bmatrix}$$

which we use to obtain the component in $\mathcal{H}_{\tilde a} = \text{span}\{\Psi_1(\tilde a), \Psi_2(\tilde a)\}^\perp$ of (8):

$$\frac{\partial\tilde\eta}{\partial t} = -iL_{\tilde\psi_E}\tilde\eta - iF_2(\tilde\psi_E,\tilde\eta) - (\mathbb{I} - M_{\tilde u})^{-1}F_3(\tilde\psi_E,\tilde\eta)$$

where F_3 is the projection of $-iF_2$ onto span$\{\frac{\partial\tilde\psi_E}{\partial a_1}, \frac{\partial\tilde\psi_E}{\partial a_2}\}$:

$$F_3(\tilde\psi_E,\tilde\eta) = \Re\langle\Psi_1(\tilde a), -iF_2(\tilde\psi_E,\tilde\eta)\rangle\cdot\frac{\partial\tilde\psi_E}{\partial a_1} + \Re\langle\Psi_2(\tilde a), -iF_2(\tilde\psi_E,\tilde\eta)\rangle\cdot\frac{\partial\tilde\psi_E}{\partial a_2} \tag{13}$$

and $\mathbb{I}-M_{\tilde u}$ is the linear operator on the two dimensional real vector space span$\{\frac{\partial\tilde\psi_E}{\partial a_1}, \frac{\partial\tilde\psi_E}{\partial a_2}\}$ whose matrix representation relative to the basis span$\{\frac{\partial\tilde\psi_E}{\partial a_1}, \frac{\partial\tilde\psi_E}{\partial a_2}\}$ is $\mathbb{I}_{\mathbb{R}^2} - M_{\tilde u}$. It is easier to switch back to the variable $\eta(t) = e^{-i\theta(t)}\tilde\eta(t)\in\mathcal{H}_a$:

$$\frac{\partial\eta}{\partial t} = -i(-\Delta+V)\eta - iDg_{\psi_E}\eta - iF_2(\psi_E,\eta) - (\mathbb{I}-M_u)^{-1}F_3(\psi_E,\eta) \tag{14}$$

where we used the equivariant symmetry and its obvious consequences for the symmetries of Dg, F_2, F_3 and M. Since by Lemma 2.2 in [12] it is sufficient to get estimates for $z(t) = P_c\eta(t)$, we now project (14) onto the continuous spectrum of $-\Delta + V$ and for M_u we switch to the notation.

$$\frac{\partial z}{\partial t} = -i(-\Delta+V)z - iP_cDg_{\psi_E}R_az - iP_cF_2(\psi_E, R_az) - P_c(\mathbb{I}-M_a[R_az])^{-1}F_3(\psi_E, R_az) \tag{15}$$

where $R_a : \mathcal{H}_0 \mapsto \mathcal{H}_a$ is the inverse of P_c restricted to $\mathcal{H}_a$, see Lemma 2.2 in [12].

Consider the initial value problem for the linear part of (15):

$$\begin{aligned} \frac{\partial \zeta}{\partial t} &= -i(-\Delta + V)\zeta - iP_cDg_{\psi_E}R_{a(t)}\zeta \\ \zeta(s) &= v \end{aligned} \tag{16}$$

and write its solution in terms of a family of operators:

$$\Omega(t,s) : \mathcal{H}_0 \mapsto \mathcal{H}_0, \quad \Omega(t,s)v = \zeta(t)$$

In [11] it has shown that such a family of operators exists. In particular $\Omega(t,s)$ satisfies certain dispersive decay estimates in weighted L^2 spaces and L^p, $p > 2$ spaces, see Theorem 4.1 and Theorem 4.2 in [11].

Then using Duhamel formula, the solution of (15) also satisfies:

$$z(t) = \Omega(t,0)z(0) - i\int_0^t \Omega(t,s)P_c[F_2(\psi_E(s), R_{a(s)}z(s)) - i(\mathbb{I}-M_{a(s)}[R_a(s)z(s)])^{-1}F_3(\psi_E(s), R_{a(s)}z(s))]ds \tag{17}$$

In order to apply the linear estimates in [11], we fix $\sigma > 2$ and $6 \leq q$, then we consider the $\varepsilon_1(q) > 0$ given by Theorem 4.1 and choose $\varepsilon_0 > 0$ in the hypothesis such that

$$\|\langle x\rangle^{4\sigma}\psi_{E(t)}(x)\|_{L^\infty} \leq \varepsilon_1, \quad \text{for all } t \in \mathbb{R} \tag{18}$$

In order to apply a contraction mapping argument for (17) we use the following Banach spaces. Recall $r > 0$ defined in Remark 2.4 in [12], then

$$Y = \{u \in L^2 \cap L^q : \sup_t (1+|t|)^{4(\frac{1}{2}-\frac{1}{q})}\|u\|_{L^q} < \infty, \sup_t \|u\|_{L^2} \leqslant r\}$$

endowed with the norm

$$\|u\|_Y = \max\{\sup_t (1+|t|)^{4(\frac{1}{2}-\frac{1}{q})}\|u\|_{L^q}, \sup_t \|u\|_{L^2}\}.$$

Consider the nonlinear operator in (17):

$$N(u) = i\int_0^t \Omega(t,s)P_c[F_2(\psi_E, u) - i(\mathbb{I} - M_a[u])^{-1}F_3(\psi_E, u)]ds$$

Lemma 3.1. *Assume* (18) *holds then,* $N : Y \to Y$ *is well defined, and locally Lipschitz, i.e. there exists* $\tilde{C} > 0$*, such that*

$$\|Nu_1 - Nu_2\|_Y \leq \tilde{C}(\|u_1\|_Y + \|u_2\|_Y)\|u_1 - u_2\|_Y.$$

Note that the Lemma gives the estimates for $z(t)$ then using Lemma 2.2 in [12] we get the estimates for $\eta(t)$ in the Theorem 3.1. Indeed, if we denote:

$$v = \Omega(t,0)z(0),$$

then

$$\|v\|_Y \leq C_0 \|z(0)\|_{L^{q'} \cap H^1},$$

where $C_0 = \max\{C, C_p\}$, see theorem 4.1 in [11]. We choose ϵ_0 in the hypotheses of theorem 3.1, such that $R = 2\|v\|_Y$ satisfies

$$Lip = 2\tilde{C}R < 1.$$

In this case the integral operator given by the right hand side of the (17):

$$K(z) = v + N(z)$$

leaves the ball $B(0,R) = \{z \in Y : \|z\|_{Y_i} \leq R\}$ invariant and it is a contraction on $B(0,R)$ with Lipschitz constant Lip. Consequently the equation (17) has a unique solution in $B(0,R)$. In particular, $z(t)$ satisfies the L^p estimates as claimed by the theorem. Then $\eta(t) = R_{a(t)}z(t)$ satisfies the L^p estimates claimed in the Theorem 3.1 by Lemma 2.2 in [12]. We now have two solutions of (17), one in $C(\mathbb{R}, H^1)$ from classical well posedness theory and one in $C(\mathbb{R}, L^2 \cap L^q)$, from the above argument. Using uniqueness and the continuous embedding of H^1 in $L^2 \cap L^q$, we infer that the solutions must coincide. Therefore, the time decaying estimates in the space Y hold also for the H^1 solution.

Proof of Lemma 3.1 Let u_1, u_2 be in the space Y. Then at each $s \in \mathbb{R}$ we have:

$$\begin{aligned} |F_2(\psi_E(s), u_1(s)) - F_2(\psi_E(s), u_2(s))| &= |g(\psi_E + u_1) - g(\psi_E + u_2) - F_1(\psi_E, u_1) + F_1(\psi_E, u_2)| \\ &\leq C(|u_1| + |u_2|)|u_1 - u_2| \end{aligned} \tag{19}$$

For a linear operator M acting on the two dimensional vector space span$\{\frac{\partial \psi_E}{\partial a_1}, \frac{\partial \psi_E}{\partial a_2}\}$, using (13) we have, for any $1 \leqslant p \leqslant \infty$:

$$\begin{aligned} \|MF_3(\psi_E, u_1)\|_{L^p} &\leq \left\| M\Re\langle \Psi_1(a), -iF_2(\psi_E, u_1)\rangle \cdot \frac{\partial \psi_E}{\partial a_1} \right\|_{L^p} + \left\| M\Re\langle \Psi_2(a), -iF_2(\psi_E, \eta)\rangle \cdot \frac{\partial \psi_E}{\partial a_2} \right\|_{L^p} \\ &\leq \|M\| \cdot |\Re\langle \Psi_1(a), -iF_2(\psi_E, u_1)\rangle| \cdot \left\| \frac{\partial \psi_E}{\partial a_1} \right\|_{L^p} + \|M\| \cdot |\Re\langle \Psi_1(a), -iF_2(\psi_E, u_1)\rangle| \cdot \left\| \frac{\partial \psi_E}{\partial a_1} \right\|_{L^p} \end{aligned}$$

where $\|M\|$ denotes the operator norm with respect to the euclidian distance in $\mathbb{R}^2$ of the representation of M with respect to the basis $\{\frac{\partial \psi_E}{\partial a_1}, \frac{\partial \psi_E}{\partial a_2}\}$. By (19) (with $u_2 = 0$), and Hölder inequality inside the L^2 scalar product we get:

$$\|MF_3(\psi_E, u_1)\|_{L^p} \leq C\|M\| \Big(\Big\|\Psi_1\Big\|_{L^q} \Big\| \frac{\partial \psi_E}{\partial a_1} \Big\|_{L^p} + \Big\|\Psi_2\Big\|_{L^q} \Big\| \frac{\partial \psi_E}{\partial a_2} \Big\|_{L^p} \Big) \Big\| |u_1|^2 \Big\|_{L^{q'}} \tag{20}$$

where the uniform bounds on $\frac{\partial \psi_E}{\partial a_j}$, $\Psi_j(a)$, $j = 1,2$ follow from the continuous dependence on scalar $a, |a(t)| \leqslant \delta_2$, $t \in \mathbb{R}$ of $\frac{\partial \psi_E}{\partial a_j} \in H^2(\mathbb{R}^3)$, $j = 1,2$, and from the definitions (10) together with the estimate

$$\Re\langle i\frac{\partial \psi_E}{\partial a_1}, \frac{\partial \psi_E}{\partial a_2}\rangle = \Re\langle -i\frac{\partial \psi_E}{\partial a_2}, \frac{\partial \psi_E}{\partial a_1}\rangle \geqslant \frac{1}{2}.$$

Using now the estimates in Remark 2.4 in [12], the matrix identity

$$(\mathbb{I}-M_a[u_1])^{-1}-(\mathbb{I}-M_a[u_2])^{-1}=(\mathbb{I}-M_a[u_1])^{-1}M_a[u_1-u_2](\mathbb{I}-M_a[u_2])^{-1}$$

the estimate (20) and again (19) we get, for any $1\leqslant p\leqslant\infty$:

$$\begin{aligned}&\|(\mathbb{I}-M_a[u_1])^{-1}F_3(\psi_E,u_1)-(\mathbb{I}-M_a[u_2])^{-1}F_3(\psi_E,u_2)\|_{L^p}\leq \\ &\leq \|[(\mathbb{I}-M_a[u_1])^{-1}-(\mathbb{I}-M_a[u_2])^{-1}]F_3(\psi_E,u_1)\|_{L^p}+\|(\mathbb{I}-M_a[u_2])^{-1}(F_3(\psi_E,u_1)-F_3(\psi_E,u_2)\|_{L^p} \\ &\leq 4C_M\|u_1-u_2\|_{L^2}C_1\|u_1^2\|_{L^{q'}}+2C_1\|(|u_1|+|u_2|)|u_1-u_2|\|_{L^{q'}}\end{aligned} \tag{21}$$

Note that $\|u_1^2\|_{L^{q'}}$ is exactly $\|(|u_1|+|u_2|)|u_1-u_2|\|_{L^{q'}}$ with $u_2=0$. So the estimates for the latter will be valid for the former provided we make $u_2=0$.

Now let us consider the difference Nu_1-Nu_2 :

$$\begin{aligned}(Nu_1-Nu_2)(t)=i\int_0^t\Omega(t,s)P_c\big[&F_2(\psi_E(s),u_1(s))-F_2(\psi_E(s),u_2(s)) \\ &-i(\mathbb{I}-M_{a(s)}[u_1(s)])^{-1}F_3(\psi_E(s),u_1(s)) \\ &+i(\mathbb{I}-M_{a(s)}[u_2(s)])^{-1}F_3(\psi_E(s),u_2(s))\big]ds\end{aligned} \tag{22}$$

- L^q **Estimate :**

  $$\|Nu_1-Nu_2\|_{L^q}\leq\int_0^t\|\Omega(t,s)\|_{L^{q'}\to L^q}C\Big(\|(|u_1|+|u_2|)|u_1-u_2|\|_{L^{q'}}+C_M\|u_1-u_2\|_{L^2}\|u_1^2\|_{L^{q'}}\Big)ds$$

  To estimate the integral observe that

  $$\begin{aligned}\|(|u_1|+|u_2|)|u_1-u_2|\|_{L^{q'}}&\leq(\|u_1\|_{L^2}+\|u_2\|_{L^2})\|u_1-u_2\|_{L^q}\\ \|u_1^2\|_{L^{q'}}&\leq\|u_1\|_{L^2}\|u_1\|_{L^q}\end{aligned}$$

  with $\frac{1}{q'}=\frac{1}{2}+\frac{\beta}{q}$. Using Theorem 4.2 in [11], we have for $u_1,\ u_2\in Y$:

  $$\begin{aligned}\|Nu_1-Nu_2\|_{L^q}&\leq\int_0^t\frac{C(q)(\|u_1\|_Y+\|u_2\|_Y)\|u_1-u_2\|_Y}{|t-s|^{4(\frac{1}{2}-\frac{1}{q})}(1+|s|)^{4(\frac{1}{2}-\frac{1}{q})}}ds\\ &\leq\frac{C(q)C_1C_2}{(1+|t|)^{4(\frac{1}{2}-\frac{1}{q})}}(\|u_1\|_Y+\|u_2\|_Y)\|u_1-u_2\|_Y\end{aligned}$$

  where $C_1=\sup_t(1+|t|)^{4(\frac{1}{2}-\frac{1}{q})}\int_0^t\frac{ds}{|t-s|^{4(\frac{1}{2}-\frac{1}{q})}(1+|s|)^{4(\frac{1}{2}-\frac{1}{q})}}<\infty$ since $6\leq q$.

- L^2 **Estimate :** To estimate L^2 norm we cannot use $L^2\to L^2$ estimate for $\Omega(t,s)$ because that would force us to control L^4 which cannot pe interpolated between L^2 and $L^q,\ 6\leq q$. We avoid this by using the decomposition:

  $$\Omega(t,s)=(T(t,s)-\tilde{T}(t,s))+(\tilde{T}(t,s)+e^{-iH(t-s)}P_c)$$

where

$$\tilde{T}(t,s) = \int_s^{\min\{t,s+1\}} e^{-iH(t-\tau)} P_c g_u(\tau) R_a e^{-iH(\tau-s)} P_c d\tau$$
$$= \int_s^{\min\{t,s+1\}} e^{iH(t-s)} P_c e^{iH(\tau-s)} P_c g_u(\tau) R_a e^{-iH(\tau-s)} P_c d\tau$$

For $T(t,s) - \tilde{T}(t,s)$ we will use $L^{q'} \to L^2$ estimates, see Theorem 4.2 in [11], while for $\tilde{T}(t,s)$ we will use duality argument with Stricharz estimates and for $e^{-iH(t-s)}P_c$ we will use Stricharz estimates $L^\infty_t L^2_x$. All in all we have:

$$\|Nu_1 - Nu_2\|_{L^2} \leq \int_0^t \|\Omega(t,s)P_c\|_{L^{p'}\to L^2} \big\| -i(\mathbb{I} - M_{a(s)}[u_1(s)])^{-1} F_3(\psi_E(s), u_1(s))$$
$$+ i(\mathbb{I} - M_{a(s)}[u_2(s)])^{-1} F_3(\psi_E(s), u_2(s)) \big\|_{L^{p'}} ds$$
$$+ \int_0^t \|T(t,s) - \tilde{T}(t,s)\|_{L^{q'}\to L^2} \|(|u_1| + |u_2|)|u_1 - u_2|\|_{L^{q'}} ds$$
$$+ \| \int_0^t e^{-iH(t-s)} P_c(|u_1| + |u_2|)|u_1 - u_2| ds\|_{L^2}$$
$$+ \| \int_0^t \tilde{T}(t,s) P_c(|u_1| + |u_2|)|u_1 - u_2| ds\|_{L^2}$$

For the first integral we use Theorem 4.2 part (i) in [11], (21) with $p = 2$ and the estimates we have already obtained for $(|u_1| + |u_2|)|u_1 - u_2|$ and $|u_1|u_1$ and similarly for the second integral we use Theorem 4.2 part (iv) in [11] the estimates we have already obtained for $(|u_1| + |u_2|)|u_1 - u_2|$. We deduce that these integrals are uniformly bounded by:

$$\tilde{C}(\|u_1\|_Y + \|u_2\|_Y)\|u_1 - u_2\|_Y.$$

For the third integral we use Stricharz estimate:

$$\sup_{t\in\mathbb{R}} \| \int_0^t e^{-iH(t-s)} P_c(|u_1|+|u_2|)|u_1 - u_2| ds\|_{L^2} \leq C_s \left(\int_{\mathbb{R}} \|(|u_1| + |u_2|)|u_1 - u_2|\|^{\gamma'}_{L^{q'}} ds \right)^{\frac{1}{\gamma'}}$$

where $\frac{1}{\gamma'} + \frac{1}{\gamma} = 1$, and $\frac{2}{\gamma} = 4(\frac{1}{2} - \frac{1}{q})$. Using again the estimates we obtained before for $(|u_1| + |u_2|)|u_1 - u_2|$ we get:

$$\|(|u_1| + |u_2|)|u_1 - u_2|\|_{L^{\gamma'}_s L^{q'}} \leq C_{11} \Big[\int_{\mathbb{R}} \frac{ds}{(1+|s|)^{8(\frac{1}{2}-\frac{1}{q})\gamma'}} \Big]^{\frac{1}{\gamma'}} (\|u_1\|_Y + \|u_2\|_Y)\|u_1 - u_2\|_Y$$
$$\leq C_{11} C_8 (\|u_1\|_Y + \|u_2\|_Y)\|u_1 - u_2\|_Y \qquad (23)$$

where $C_8 = \int_{\mathbb{R}} \frac{ds}{(1+|s|)^{8(\frac{1}{2}-\frac{1}{q})\gamma'}} ds < \infty$ since $8(\frac{1}{2} - \frac{1}{q})\gamma' > 1$.

Now for the last integral consider

$$\tilde{A} = \| \int_0^t \tilde{T}(t,s) A ds\|_{L^2}$$

with $A = (|u_1| + |u_2|)|u_1 - u_2|$. Fix $t \geq 0$. By duality

$$\begin{aligned}
\tilde{A} &= \sup_{\|\tilde{v}\|_{L^2}=1} \left|\langle \tilde{v}, \int_0^t \tilde{T}(t,s)Ads\rangle\right| \\
&\leq \sup_{\|\tilde{v}\|_{L^2}=1} \int_0^t \left|\langle e^{iH(t-s)}P_c\tilde{v}, \int_s^{\min\{t,s+1\}} e^{iH(\tau-s)}P_c g_u(\tau)R_a e^{-iH(\tau-s)}P_c A d\tau\rangle\right| ds \\
&\leq \sup_{\|\tilde{v}\|_{L^2}=1} \int_0^t \|e^{iH(t-s)}P_c\tilde{v}\|_{L^p} \int_s^{\min\{t,s+1\}} \|e^{iH(\tau-s)}P_c g_u(\tau)R_a e^{-iH(\tau-s)}P_c\|_{L^{p'}\to L^{p'}} d\tau \|A\|_{L^{p'}} ds \\
&\leq \sup_{\|\tilde{v}\|_{L^2}=1} \int_0^t \|e^{iH(t-s)}P_c\tilde{v}\|_{L^p} \sup_{\tau\in[s,s+1]} \|\widehat{g_u(\tau)}\|_{L^1}\|A\|_{L^{p'}} ds
\end{aligned}$$

Note that

$$\|e^{iH(t-s)}\tilde{v}\|_{L^\gamma L^q_x} \leq C_s\|\tilde{v}\|_{L^2} = C_s$$

by Stricharz estimate and using (23) for $\|A\|_{L^{\gamma'}L^{q'}}$ we get the required estimates for $\tilde{A}$.

The L^2 estimates are now complete and the proof of Lemma 3.1 is finished. □

We now finish the proof of Theorem 3.1 by analyzing the dynamics on the center manifold and showing it converges to a ground state. From equation (12) we have

$$|\tilde{a}'| = C\sqrt{F_{21}^2 + F_{22}^2} = b(t)$$

and

$$\left|[a(t)e^{i\int_0^t E(s)ds}]'\right| = b(t)$$

Since $b(t) = C\sqrt{F_{21}^2 + F_{22}^2}$, and

$$F_{21} \leq \left\|\frac{\partial\psi_E}{\partial a_2}\right\|_{L^{p_2}} \|A\|_{L^{q'}} \leq C\|\eta\|_{L^q}^2$$

$$F_{22} \leq \left\|\frac{\partial\psi_E}{\partial a_1}\right\|_{L^{p_2}} \|A\|_{L^{q'}} \leq C\|\eta\|_{L^q}^2$$

we get $0 \leqslant b(t) \leqslant C(1+|t|)^{1+\delta}$ for some $\delta > 0$, in the Theorem 3.1. Then, for any $\varepsilon > 0$ we have

$$\left|a(t)e^{i\int_0^t E(s)ds} - a(t')e^{i\int_0^{t'} E(s)ds}\right| \leqslant \int_{t'}^t b(s)ds < \varepsilon \tag{24}$$

for $t,\ t'$ sufficiently large respectively sufficiently small. Therefore $a(t)e^{i\int_0^t E(s)ds}$ has a limit when $t \to \pm\infty$. This means

$$e^{i\int_0^t E(s)ds}\psi_E = a(t)e^{i\int_0^t E(s)ds}\psi_0 + e^{i\int_0^t E(s)ds}h(a(t)) = a(t)e^{i\int_0^t E(s)ds}\psi_0 + h(a(t)e^{i\int_0^t E(s)ds}) \to \psi_{E_{\pm\infty}}$$

Above we used $h(e^{i\theta}a) = e^{i\theta}h(a)$, see Proposition 2.1 in [12]. In addition $|a(t)| \to a_\pm$ as $t \to \pm$ at a rate $|t|^{-\delta}$. Since $E(s) = E(|a(s)|$ is C^1 in $|a|$ on $|a| \leqslant \delta_2$, we deduce $|E(\pm s) - E_\pm| \leqslant C(1+s)^{-\delta}$ for $s \geqslant 0$ and some constant $C > 0$. If we denote

$$\theta(\pm t) = \frac{1}{\pm t}\int_0^{\pm t} E(s) - E_\pm ds, \qquad t \geqslant 0$$

then $\lim_{|t|\to\infty} \theta(t) = 0$ and

$$\lim_{t\to\pm} e^{it(E_\pm - \theta(t))}\psi_E(t) = \psi_{E_\pm}.$$

This finishes the proof of Theorem 3.1. □

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