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On a Boundary Value Problem For a Second Order ODE

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Abstract

We investigate the existence of solutions to a boundary value problem for a second order ordinary differential equation (ODE) over an unbounded interval. The conclusions are useful in studying certain reaction-diffusion equations via the comparison method.

Key-words: *ordinary differential equation, boundary value problem, fixed point theory.*

1. Introduction

This paper is concerned with the existence of a positive solution to the following boundary value problem

$$\begin{aligned} & u'' + a(t)u = 0, t \geq t_0 \geq 1, \\ & u'(t) - \frac{q(t)}{t}u(t) \leq 0, t \geq t_0, \\ & u(t) = o(t) \text{ as } t \to +\infty, \end{aligned} \tag{1}$$

where $a : [t_0, +\infty) \to [0, +\infty), q : [t_0, +\infty) \to (0,1]$ are assumed continuous.

The interest in studying the *q-problem* (1) comes from an investigation of the existence and decay rates of the positive, vanishing at $+\infty$ classical solutions to the reaction-diffusion equations

$$\Delta u + f(x,u) + g(|x|)\, x \cdot \nabla u = 0, \qquad |x| \geq R > 0, \qquad x \in IR^n.$$

For an account of recent literature on this topic, we refer to the studies [1]-[3], [5], [6], [8].

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Several technical conditions are imposed on the coefficient $q(t)$, namely

$$q(t) = o(1) \text{ when } t \to +\infty$$

$$\frac{t}{q(t)} \int_t^{+\infty} \left[\frac{q(s)}{s} \right]^2 \exp\left(\int_{t_0}^{s} \frac{q(\tau)}{\tau} d\tau \right) ds \le \sigma < \frac{1}{2}, t \ge t_0, \qquad (2)$$

$$\exp\left(\int_{t_0}^{t} \frac{q(\tau)}{\tau} d\tau \right) = o(t) \text{ when } t \to +\infty.$$

In can be easily seen that the function $q(t) = (\ln t)^{-1}, t \ge t_0 \ge e^2$, verifies the hypotheses (2). See the paper [3].

Our result, based on application of a fixed point technique, improves upon the conclusions of the investigations [3], [6].

2. Positive solution to the q-problem

Consider the linear ordinary differential equation of (1),

$$u'' + a(t)u = 0, t \ge t_0 \ge 1, \qquad (3)$$

and assume that it possesses a positive solution $u(t)$.

Hartman's non-oscillation theorem [7] implies that

$$\int_{t_0}^{+\infty} a(t)dt < +\infty. \qquad (4)$$

Further, since $u''(t) \le 0, t \ge t_0$, there exists $\lim_{t \to +\infty} u'(t) = l \in [0, u'(t_0)]$. Also, we have

$$u'(t) = l + \int_t^{+\infty} a(s)u(s)ds, \qquad t \ge t_0. \qquad (5)$$

The case $l > 0$ has been discussed in [3], [5], [6]. In this circumstance, we deduce from (5) that $u(t) = l.t + o(t)$ for $t \to +\infty$ and

$$\int_{t_0}^{+\infty} ta(t)dt < +\infty. \qquad (6)$$

We shall consider hereafter the case $l = 0$.

By means of an integration by parts, relation (5) yields the system

$$u'(t) = v(t),$$

$$v(t) = \int_t^{+\infty} v(s) \left(\int_s^{+\infty} a(\tau)d\tau \right) ds + \left(\int_t^{+\infty} a(s)ds \right) \left(u(t_0) + \int_{t_0}^{t} v(s)ds \right) \qquad (7)$$

in $[t_0, +\infty)$.

Following (7), the positive solution $u(t)$ of equation (3) verifies the inequality

$$u'(t) \geq \left(t \int_t^{+\infty} a(s)ds \right) \frac{u(t)}{t}, t \geq t_0. \tag{8}$$

This means that, if $u(t)$ is a solution of q-problem (1), the function $a(t)$ must be confined by

$$\frac{t}{q(t)} \int_t^{+\infty} a(s)ds \leq 1. \tag{9}$$

Theorem. Assume that $q(t)$ verifies (2) and $a(t)$ verifies (4), (9).

(i) There exists $\varepsilon \in (0,1)$ such that

$$u'' + a(t)u = 0, t \geq t_0 \geq 1,$$

$$u'(t) - (1+\varepsilon)\frac{q(t)}{t} u(t) \leq 0, t \geq t_0, \tag{10}$$

$$u(t) = o(t) \text{ as } t \to +\infty.$$

(ii) If

$$\int_{t_0}^{+\infty} \frac{q(s)}{s} ds < +\infty \tag{11}$$

then $\int_{t_0}^{+\infty} sa(s)ds < +\infty$ and the solution of (10) from (i) is bounded.

Proof. *The first step.* Fix the numbers $c, c_1 > 0, \varepsilon \in (0,1)$. Assume that it exists the continuous function $p : [t_0, +\infty) \to (0, +\infty)$ such that

$$c_1 \left[\int_t^{+\infty} \frac{q(s)}{sp(s)} ds + \frac{q(t)}{t} \int_{t_0}^{t} \frac{ds}{p(s)} \right] + c\frac{q(t)}{t} \leq \frac{c_1}{p(t)} \tag{12}$$

and

$$\frac{t}{q(t)} \int_t^{+\infty} \frac{q(s)}{sp(s)} ds \leq \varepsilon \frac{c}{c_1} \tag{13}$$

for all $t \geq t_0$. Also,

$$\int_{t_0}^{t} \frac{ds}{p(s)} = o(t) \text{ when } t \to +\infty. \tag{14}$$

Set now $C = \{v \in C([t_0, +\infty), IR) : 0 \leq p(t)v(t) \leq c_1, t \geq t_0\}$. A partial order on C is given by the usual pointwise order "$\leq$", that is, we say that $v_1 \leq v_2$ if and only if $v_1(t) \leq v_2(t)$ for all $t \geq t_0$.

Introduce the operator $T : C \to C([t_0, +\infty), IR)$ by (recall (7))

$$T(v)(t) = \int_t^{+\infty} v(s) \left(\int_s^{+\infty} a(\tau)d\tau \right) ds + \left(\int_t^{+\infty} a(s)ds \right) \left(c + \int_{t_0}^{t} v(s)ds \right),$$

where $v \in C$ and $t \ge t_0$.

Given $v \in C$, we have the next estimates

$$\int_t^{+\infty} v(s)\left(\int_s^{+\infty} a(\tau)d\tau\right)ds + \left(\int_t^{+\infty} a(s)ds\right)\left(\int_{t_0}^{t} v(s)ds\right) \le \left[\int_t^{+\infty} \frac{q(s)}{sp(s)}ds + \frac{q(t)}{t}\int_{t_0}^{t}\frac{ds}{p(s)}\right]c_1$$

and respectively

$$T(v)(t) \le \left[\int_t^{+\infty} \frac{q(s)}{sp(s)}ds + \frac{q(t)}{t}\int_{t_0}^{t}\frac{ds}{p(s)}\right]c_1 + \frac{q(t)}{t}c \le \frac{c_1}{p(t)}, t \ge t_0,$$

by means of (12). Accordingly, $T(C) \subseteq C$.

Since the functional coefficient $a(t)$ is nonnegative, the application T is isotone, that is, $T(v_1) \le T(v_2)$ whenever $v_1 \le v_2$, and it satisfies $0 \le T(0)$. By application of the Knaster-Tarski fixed point theorem [4, p. 14], T has a fixed point in C, denoted v_0.

The solution of the "ε-perturbed" q-problem (10) (or, properly speaking $(1+\varepsilon)q$-problem (10)) is given by the formula $u(t) = c + \int_{t_0}^{t} v_0(s)ds$ for all $t \ge t_0$.

In fact, we have

$$u'(t) = v_0(t) = T(v_0)(t) \le \int_t^{+\infty} \frac{q(s)}{sp(s)}ds\,c_1 + \frac{q(t)}{t}u(t) \le \varepsilon q(t)\frac{c}{t} + q(t)\frac{u(t)}{t}$$

$$\le (1+\varepsilon)q(t).\frac{u(t)}{t},$$

by means of (13), throughout $[t_0, +\infty)$ and respectively (recall (14))

$$c \le u(t) \le c + \int_{t_0}^{t}\frac{ds}{p(s)} = o(t) \text{ when } t \to +\infty.$$

The second step. Conditions (12)-(14), though essential for establishing the existence of a positive solution to (10), are quite complicated. Several simplifications will be performed here in order to make the preceding approach workable.

Fix the number $\eta = \frac{c_1}{(1+\varepsilon)c}$. Further, take

$$p(t) = \frac{\eta . t\, G(t)}{q(t)} \qquad G(t) = \exp\left(-\int_{t_0}^{t}\frac{q(\tau)}{\tau}d\tau\right)$$

in $[t_0, +\infty)$. Then,

$$\frac{q(t)}{t}\int_{t_0}^{t}\frac{ds}{p(s)} = \frac{1-G(t)}{p(t)}, \qquad (1+\varepsilon)\frac{c}{c_1} = \frac{tG(t)}{q(t)p(t)}$$

and

$$c_1\left[\int_t^{+\infty}\frac{q(s)}{sp(s)}ds+\frac{q(t)}{t}\int_{t_0}^{t}\frac{ds}{p(s)}\right]+c\frac{q(t)}{t}=\left[c_1\int_t^{+\infty}\frac{q(s)}{sp(s)}ds+c\frac{q(t)}{t}\right]+c_1\frac{q(t)}{t}\int_{t_0}^{t}\frac{ds}{p(s)}$$

$$\leq(1+\varepsilon)c\cdot\frac{q(t)}{t}+c_1\frac{q(t)}{t}\int_{t_0}^{t}\frac{ds}{p(s)}=c_1\left[\frac{G(t)}{p(t)}+\frac{1-G(t)}{p(t)}\right]=\frac{c_1}{p(t)},t\geq t_0.$$

Also, (13) is a consequence of the second condition (2) provided that we take $\varepsilon\in(0,1)$ such that $\sigma<\frac{\varepsilon}{1+\varepsilon}$.

Finally, the restriction (14) reads as

$$\int_{t_0}^{t}\frac{ds}{p(s)}=\eta\left[\frac{1}{G(t)}-\frac{1}{G(t_0)}\right]=o(t)\quad\text{as}\quad t\to+\infty,$$

by means of the third condition (2).

The proof of (i) is complete.

Assume now that (11) holds true. Thus, (6) follows directly from (9). Notice further that (2) yields

$$\frac{t}{q(t)}\int_t^{+\infty}\left[\frac{q(s)}{s}\right]^2ds\leq\sigma<\frac{1}{2},t\geq t_0. \tag{15}$$

Fix again $c,c_1>0$. By eventually increasing $t_0\geq1$, introduce the numbers $\alpha,\theta\in(0,1)$ in order to have

$$\frac{c_1}{\alpha}\int_{t_0}^{+\infty}\frac{q(s)}{s}ds+c<(\theta-\sigma)\frac{c_1}{\alpha},\quad \sigma\leq\alpha\,\varepsilon.\,\frac{c}{c_1} \tag{16}$$

This leads us, via (15), to an improved variant of (12) for $p(t)=\alpha\frac{t}{q(t)}$ in $[t_0,+\infty)$. Precisely, we have

$$c_1\left[\int_t^{+\infty}\frac{q(s)}{sp(s)}ds+\frac{q(t)}{t}\int_{t_0}^{t}\frac{ds}{p(s)}\right]+c\frac{q(t)}{t}\leq\theta\frac{c_1}{p(t)} \tag{17}$$

for all $t\geq t_0$. The second inequality in (16) implies (13). The verification of (14) is trivial.

If the set C is endowed with the metric

$$d(v_1,v_2)=\sup_{t\geq t_0}\{p(t)\,|\,v_1(t)-v_2(t)\,|\},\qquad v_1,v_2\in C,$$

then $S=(C,d)$ becomes a complete metric space.

Now, for any $v_1,v_2\in C$, we have the estimate

$$|T(v_1)(t)-T(v_2)(t)|\leq\left[\int_t^{+\infty}\frac{q(s)}{sp(s)}ds+\frac{q(t)}{t}\int_{t_0}^{t}\frac{ds}{p(s)}\right]\cdot d(v_1,v_2)\leq\frac{\theta}{p(t)}\cdot d(v_1,v_2) \tag{18}$$

for $t\geq t_0$.

Finally, is a contraction of coefficient by means of (17), (18). Its fixed point in being denoted with , the proof is completed by following the discussion from (i).

3. Open problem

We remark that, in the case of from [3], we have the identity

$$\frac{t}{q(t)}\int_t^{+\infty}\left[\frac{q(s)}{s}\right]^2\exp\left(\int_{t_0}^{s}\frac{q(\tau)}{\tau}d\tau\right)ds=\frac{t}{q(t)}\int_t^{+\infty}\frac{q(t_0)q(s)}{s^2}ds, t\geq t_0. \tag{19}$$

It is still unknown how to solve the q-problem (1) *via the integral operator provided by (7)* when the second of hypotheses (2) is replaced by the right-hand member of (19) or, ideally, by (recall (9))

$$\frac{t}{q(t)}\int_t^{+\infty}a(s)ds\leq 1-\frac{t}{q(t)}\int_t^{+\infty}\left[\frac{q(s)}{s}\right]^2ds,\quad t\geq t_0.$$

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