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Numerical Solution of Volterra Integral Equations Using the Chebyshev-Collocation Spectral Methods

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Abstract- The main purpose of this paper is to submit a new numerical approach for the Volterra integral equations based on a spectral method. The Chebyshev-collocation spectral method is proposed to solve the Volterra integral equations of the second kind and then convergence analysis of proposed method is discussed. Numerical examples show that the approximate solutions have a good degree of accuracy.

Keywords- *Second kind Volterra integral equations, collocation spectral method , Chebyshev polynomials.*

I. INTRODUCTION

This paper consider the second kind Volterra integral equations

$$y(t)+\int_0^t R(t,s)y(s)ds=f(t) \quad ,t\in[0,T] \qquad (1.1)$$

Where the source function $f(t)$ and the kernel function $R(t,s)$ are given, and $y(t)$ is the unknown function.
In order that the solutions of (1.1) be sufficiently smooth, it is necessary to use very high-order numerical methods such as spectral methods for approximating the solutions.
For this purpose, we transfer the problem (1.1) to an equivalent problem defined in $[-1,1]$. First using the change of variables $t=\frac{T(1+x)}{2}$, $x=\frac{2t}{T-1}$ the Volterra equation (1.1) rewrite as follows:

$$u(x)+\int_0^{T(1+x)/2} K\left(\frac{T(1+x)}{2},s\right)y(s)ds=g(x) \qquad (1.2)$$

Where

$$g(x)=f\left(\tfrac{T}{2}(1+x)\right)\ ,\ \ u(x)=y\left(\tfrac{T}{2}(1+x)\right)\ ,\ x\in[-1,1].$$

Now we use the linear transformation $s=\frac{T(1+\tau)}{2}$, $\tau\in[-1,x]$ to transfer the integral interval $[0,T(1+x)/2]$ to the interval $[-1,x]$.
Then the equation (1.2) become:

$$u(x)+\int_{-1}^{x} K(x,\tau)u(\tau)d\tau=g(x) \qquad ,x\in[-1,1] \qquad (1.3)$$

where

$$K(x,\tau)=\frac{T}{2}R\left(\frac{T}{2}(1+x),\frac{T}{2}(1+\tau)\right).$$

In this paper provide Chebyshev-collocation spectral method and its convergence analysis. This method has higher accuracy degree relative to Legendre-collocation spectral method.
This paper is organized as follows:
In section 2, Chebyshev-collocation spectral method is expressed for the Volterra integral equations of second kind . In section 3, some useful lemmas for the convergence analysis are proposed. In section 4, the convergence analysis in both L^2 and L^∞ spaces is shown. In section 5, numerical results are presented to verify the convergence obtained in section 4.

II. CHEBYSHEV-COLLOCATION METHOD

We will consider the second kind linear Volterra integral equations as:

$$u(x)+\int_{-1}^{x} K(x,s)u(s)ds=g(x) \qquad ,x\in[-1,1] \qquad (2.1)$$

Also we consider the $(N+1)$ -point collocation points as the set of Gauss, or Gauss-Radau, or Gauss-Lobatto points, $\{x_i\}_{i=0}^N$.
By using these points the equation (2.1) can be written as:

$$u(x_i)+\int_{-1}^{x_i} K(x_i,s)u(s)ds=g(x_i) \qquad ,0\le i\le N \qquad (2.2)$$

For calculating the integral, we use Gauss integration formula as:

$$\int_a^b f(x)\omega(x)dx\cong\sum_{i=0}^{N} f(x_i)\omega_i. \qquad (2.3)$$

First we transfer the integral interval to interval $[-1,1]$, by using the following linear transformation:

$$s=s(x,\theta)=\frac{1+x}{2}\theta+\frac{x-1}{2} \qquad ,-1\le\theta\le 1\ . \qquad (2.4)$$

Then the equation (2.2) is expressed as follows:

$$u(x_i)+\frac{1+x_i}{2}\int_{-1}^{1} K\big(x_i,s(x_i,\theta)\big)u\big(s(x_i,\theta)\big)\sqrt{1-\theta^2}\,\omega(\theta)d\theta = g(x_i) \qquad ,0\le i\le N \qquad (2.5)$$

where $\omega(\theta)$ indicates the Chebyshev weight.
Now by using $(N+1)$-point Gauss integration formula we get:

$$u(x_i)+\frac{1+x_i}{2}\sum_{j=0}^{N}K\big(x_i,s(x_i,\theta_j)\big)u\big(s(x_i,\theta_j)\big)\sqrt{1-\theta_j^2}\,\omega_j = g(x_i) \quad ,0\le i\le N \ , \qquad (2.6)$$

where the set $\{\theta_j\}_{j=0}^{N}$ coincide with the collocation points $\{x_j\}_{j=0}^{N}$.

Let $u\big(s(x_i,\theta_j)\big)$ be shown as u_i, then by using chebyshev interpolation polynomials, the expansion u is as:

$$u(\sigma)\approx\sum_{j=0}^{N}u_jF_j(\sigma)\ , \qquad (2.7)$$

where F_j is the j-th chebyshev basis function and

$$F_j(s)=\sum_{p=0}^{N}\alpha_{p,j}T_p(s)\ , \qquad (2.8)$$

where

$$\alpha_{p,j}=\frac{1}{\gamma_p}\sum_{i=0}^{N}T_p(x_i)F_j(x_i)\omega_i=\frac{T_p(x_j)\omega_j}{\gamma_p}\ , \quad p=0,1,2,\dots \qquad (2.9)$$

And

$$\gamma_p=\frac{\pi}{2}C_p \quad , \quad \forall p<N \qquad (2.10)$$

$$C_p=\begin{cases}2 & , \quad k=0\\ 1 & , \quad k\ge 1\end{cases} \qquad (2.11)$$

$$\gamma_N=\begin{cases}\frac{\pi}{2} & \text{for Gauss and Gauss}-\text{Radau formulas}\ ,\\ \pi & \text{for the Gauss}-\text{Lobatto formula}\end{cases} \qquad (2.12)$$

Combining equations (2.6) and (2.7) yields:

$$u_i+\frac{1+x_i}{2}\sum_{j=0}^{N}u_j\left(\sum_{p=0}^{N}K\left(x_i,s(x_i,\theta_p)\right)F_j\left(s(x_i,\theta_p)\right)\sqrt{1-\theta_p^2}\,\omega_p\right)=g(x_i)\quad ,0\le i\le N\ . \qquad (2.13)$$

We obtain $\{u_i\}_{i=0}^{N}$ by solving the above system.

III. ERROR ESTIMATES

Lemma 3.1 . ([3]. Gauss integration error)
Assume that a $(N+1)$-point Gauss, or Gauss-Radau, or Gauss-Lobatto integration formula relative to the Chebyshev weight is used to integrate the product $u\emptyset$, where $u\in H_\omega^m(I)$ with $I:=(-1,1)$ for some $m\ge 1$ and $\emptyset\in p_N$. Then will exist a constant C independent of N such that:

$$\left|\int_{-1}^{1}u(x)\emptyset(x)\omega(x)dx-(u,\emptyset)_N\right|\le CN^{-m}|u|_{H_\omega^{m,N}(I)}\|\emptyset\|_{L_\omega^2(I)} \quad (3.1)$$

where

$$|u|_{H_\omega^{m,N}(I)}=\Big(\sum_{j=\min(m,N+1)}^{m}\big\|u^{(j)}\big\|_{L_\omega^2(I)}^2\Big)^{1/2}\ , \qquad (3.2)$$

$$(u,\emptyset)_N=\sum_{j=0}^{N}\omega_ju(x_j)\emptyset(x_j) \qquad (3.3)$$

Lemma 3.2. ([3], Estimates for the interpolation error)
Assume that $u\in H_\omega^m(I)$ and denote $I_Nu\in\mathbb{P}_N$ its interpolation polynomial in the $(N+1)$-point Gauss, or Gauss-Radau, or Gauss-Lobatto points $\{x_j\}_{j=0}^{N}$, namely,

$$I_Nu=\sum_{i=0}^{N}u(x_i)F_i(x) \qquad (3.4)$$

Then the following estimates are obtained

$$\|u-I_Nu\|_{L_\omega^2(I)}\le CN^{-m}|u|_{H_\omega^{m,N}(I)} \qquad (3.5)$$

$$\|u-I_Nu\|_{L^\infty(I)}\le CN^{\frac{1}{2}-m}|u|_{H_\omega^{m,N}(I)} \qquad (3.6)$$

$$\|u-I_Nu\|_{H_\omega^l(I)}\le CN^{2l-m}|u|_{H_\omega^{m,N}(I)} \quad ,0\le l\le m \qquad (3.7)$$

Lemma 3.3. ([9] Lebesgue constant for the Chebyshev series)
Assume that $F_j(x)$ is the j-th Chebyshev interpolation polynomials in the Gauss, or Gauss-Radau, or Gauss-Lobatto points. Then

$$\max_{x\in I}\sum_{j=0}^{N}\left|F_j(x)\right|=A_N(T) \qquad (3.8)$$

where

$$A_N(T)=\frac{2}{\pi}\log N+\frac{2}{\pi}\left(\gamma+\log\frac{8}{\pi}\right)+\alpha_N \quad ,0<\alpha_N<\frac{\pi}{72N^2} \quad ,N\ge 1.$$

Lemma 3.4. ([8], Gronwall inequality)
If a non-negative integrable function $E(t)$ satisfies

$$E(t)\le C_1\int_{-1}^{t}E(s)ds+G(t) \qquad ,-1\le t\le 1 \qquad (3.9)$$

Where $G(t)$ is an integrable function, then

$$\|E\|_{L_\omega^p(I)}\le C\|G\|_{L_\omega^p(I)} \qquad ,1\le p\le\infty \qquad (3.10)$$

IV. CONVERGENCE ANALYSIS

In this section, we will investigate the convergence analysis in both L^2 and L^∞ spaces.

4.1. Error analysis in L^2 space

Theorem 4.1. Let u be the exact solution of the Volterra equation (2.1) and assume that

$$U(x)=\sum_{j=0}^{N}u_jF_j(x) \qquad (4.1)$$

Where u_j is given by (2.13) and $F_j(x)$ is the j-th Chebyshev basis function associated with the Gauss-points $\{x_j\}_{j=0}^{N}$. If $u\in H_\omega^m(I)$ and $e(x)=U(x)-u(x)$, then for $m\ge 1$ the fallowing formula be obtained:

$$\|e(x)\|_{L_\omega^2(I)}\le CN^{-(2+m)}\max_{x\in I}\left|K\big(x,s(x,.)\big)\right|_{H_\omega^{m,N}(I)}\|u\|_{L_\omega^2(I)}+CN^{-m}|u|_{H_\omega^{m,N}(I)}$$

provided that N is sufficiently large, and C is a constant independent of N.

Proof:
We rewrite the equation (2.1) as follows:

$$u_i + \frac{1+x_i}{2}\sum_{p=0}^{N} K\left(x_i, s(x_i, \theta_p)\right)\omega_p \sum_{j=0}^{N} u_j F_j\left(s(x_i, \theta_p)\right) = g(x_i) \quad , 0 \le i \le N \quad (4.1)$$

Then by using the equation (4.1) is written:

$$u_i + \frac{1+x_i}{2}\sum_{p=0}^{N} K\left(x_i, s(x_i, \theta_p)\right)\omega_p U\left(s(x_i, \theta_p)\right) = g(x_i) \ , 0 \le i \le N. \quad (4.2)$$

From the equation (3.3) we have:

$$\left(K(x,s), \emptyset(s)\right)_{N,s} = \sum_{j=0}^{N} K\left(x, s(x, \theta_j)\right)\emptyset\left(s(x, \theta_j)\right)\omega_j. \quad (4.3)$$

Now the equation (4.2) can be written as:

$$u_i + \frac{1+x_i}{2}\left(K(x_i, s), U(s)\right)_{N,s} = g(x_i) \quad , 0 \le i \le N. \quad (4.4)$$

We define:

$$J_1(x) = \frac{1+x}{2}\int_{-1}^{1} K\left(x, s(x,\theta)\right)U\left(s(x,\theta)\right)\omega\left(s(x,\theta)\right)d\theta - \frac{1+x}{2}\left(K(x,s), U(s)\right)_{N,s} \ . \quad (4.5)$$

Then

$$J_1(x_i) = \frac{1+x_i}{2}\int_{-1}^{1} K\left(x_i, s(x_i,\theta)\right)U\left(s(x_i,\theta)\right)\omega\left(s(x_i,\theta)\right)d\theta - \frac{1+x_i}{2}\left(K(x_i,s), U(s)\right)_{N,s} \quad (4.6)$$

Then we have:

$$\frac{1+x_i}{2}\left(K(x_i,s), U(s)\right)_{N,s} = \frac{1+x_i}{2}\int_{-1}^{1} K\left(x_i, s(x_i,\theta)\right)U\left(s(x_i,\theta)\right)\omega\left(s(x_i,\theta)\right)d\theta - J_1(x_i)$$

So the equation (4.4) can be written as:

$$u_i + \frac{1+x_i}{2}\int_{-1}^{1} K\left(x_i, s(x_i,\theta)\right)U\left(s(x_i,\theta)\right)\omega\left(s(x_i,\theta)\right)d\theta - J_1(x_i) = g(x_i) \ .$$

Then

$$u_i + \frac{1+x_i}{2}\int_{-1}^{1} K\left(x_i, s(x_i,\theta)\right)U\left(s(x_i,\theta)\right)\omega\left(s(x_i,\theta)\right)d\theta = g(x_i) + J_1(x_i) \ . \quad (4.7)$$

By using the equation (2.4), the equation (4.7) can be written as follows:

$$u_i + \int_{-1}^{x_i} K(x_i, s)U(s)ds = g(x_i) + J_1(x_i) \ , 0 \le i \le N \quad (4.8)$$

Since

$$U(x) = e(x) + u(x) \quad (4.9)$$

Then the equation (4.8) can be written as:

$$u_i + \int_{-1}^{x_i} K(x_i, s)\left(e(s) + u(s)\right)ds = g(x_i) + J_1(x_i) \ , 0 \le i \le N$$

Then

$$u_i + \int_{-1}^{x_i} K(x_i, s)e(s)ds \int_{-1}^{x_i} K(x_i, s)u(s)ds = g(x_i) + J_1(x_i) \ , 0 \le i \le N$$

By multiplying $F_i(x)$ in the above equation and summing up for $i = 0, \dots, N$ the following equation is obtained:

$$\sum_{i=0}^{N} u_i F_i(x) + \sum_{i=0}^{N}\int_{-1}^{x_i} K(x_i, s)e(s)ds \, . F_i(x) + \sum_{i=0}^{N}\int_{-1}^{x_i} K(x_i, s)u(s)ds \, . F_i(x) = \sum_{i=0}^{N} g(x_i) \, . F_i(x) + \sum_{i=0}^{N} J_1(x_i) \, . F_i(x)$$

By using the equations (4.1) and (3.4) we have:

$$U(x) + I_N\left(\int_{-1}^{x} K(x,s)e(s)ds\right) + I_N\left(\int_{-1}^{x} K(x,s)u(s)ds\right) = I_N g + I_N(J_1) \quad (4.10)$$

From the equation (2.1) we have:

$g(x) - u(x) = \int_{-1}^{x} K(x,s)e(s)ds.$

Then the equation (4.10) can be written as:

$$U(x) + I_N\left(\int_{-1}^{x} K(x,s)e(s)ds\right) + I_N\left(g(x) - u(x)\right) = I_N g + I_N(J_1).$$

Then

$$U(x) + I_N\left(\int_{-1}^{x} K(x,s)e(s)ds\right) + I_N g(x) - I_N u(x) = I_N g + I_N(J_1)$$

This equation by using the equation (4.9) can be expressed as follows:

$$e(x) + u(x) + I_N\left(\int_{-1}^{x} K(x,s)e(s)ds\right) + I_N g(x) - I_N u(x) = I_N g + I_N(J_1)$$

Then

$$e(x) + u(x) + I_N\left(\int_{-1}^{x} K(x,s)e(s)ds\right) - I_N u = I_N(J_1)$$

Then result will be:

$$e(x) + (u - I_N u)(x) + I_N\left(\int_{-1}^{x} K(x,s)e(s)ds\right) = I_N(J_1) \quad (4.11)$$

Now we define:

$$J_2(x) = I_N u(x) - u(x) \quad (4.12)$$

And

$$J_3(x) = \int_{-1}^{x} K(x,s)e(s)ds - I_N\left(\int_{-1}^{x} K(x,s)e(s)ds\right) \quad (4.13)$$

Then the equation (4.13) can be written as:

$$I_N\left(\int_{-1}^{x} K(x,s)e(s)ds\right) = \int_{-1}^{x} K(x,s)e(s)ds - J_3(x) \quad (4.14)$$

By using the equations (4.12) and (4.14), the equation (4.11) is expressed as follows:

$$e(x) - J_2(x) + \int_{-1}^{x} K(x,s)e(s)ds - J_3(x) = I_N(J_1)$$

Then

$$e(x) + \int_{-1}^{x} K(x,s)e(s)ds = I_N(J_1) + J_2(x) + J_3(x). \quad (4.15)$$

So we have:

$$e(x) \le \int_{-1}^{x} e(s)ds + \left(I_N(J_1) + J_2(x) + J_3(x)\right).$$

From the Lemma 3.4 is obtained:

$$\|e(x)\|_{L^2_\omega(I)} \le C\|I_N(J_1)+J_2(x)+J_3(x)\|_{L^2_\omega(I)}$$
$$\le C\left(\|I_N(J_1)\|_{L^2_\omega(I)}+\|J_2(x)\|_{L^2_\omega(I)}+\|J_3(x)\|_{L^2_\omega(I)}\right) . \quad (4.16)$$

From the equation (3.4) we have:

$I_N(J_1)=\sum_{i=0}^{N} J_1(x_i)F_i(x)$.

By using the L^2 norm the following equation is obtained:

$$\|I_N(J_1)\|_{L^2_\omega(I)} = \left\|\sum_{i=0}^{N} J_1(x_i)F_i(x)\right\|_{L^2_\omega(I)} \le \sum_{i=0}^{N}|J_1(x_i)|\,|F_i(x)| \quad (4.17)$$

By using the equation (4.6) is written:

$$J_1(x_i)=\frac{1+x_i}{2}\,(\int_{-1}^{1}K(x_i,s(x_i,\theta))U(s(x_i,\theta))\omega(s(x_i,\theta))d\theta - (K(x_i,s),U(s))_{N,s}).$$

Then

$$|J_1(x_i)| = \left|\frac{1+x_i}{2}\,(\int_{-1}^{1}K(x_i,s(x_i,\theta))U(s(x_i,\theta))\omega(s(x_i,\theta))d\theta - (K(x_i,s),U(s))_{N,s})\right|$$

$$=\left|\frac{1+x_i}{2}\right|\left|\int_{-1}^{1}K(x_i,s(x_i,\theta))U(s(x_i,\theta))\omega(s(x_i,\theta))d\theta - (K(x_i,s),U(s))_{N,s}\right|.$$

From the Lemma 3.1 is obtained:

$$|J_1(x_i)| \le CN^{-m}\left|K(x_i,s(x_i,.))\right|_{H^{m,N}_\omega(I)}\|U\|_{L^2_\omega(I)} \quad (4.18)$$

Then

$$|J_1(x_i)| \quad \le CN^{-m}\max_{x\in I}\left|K(x,s(x,.))\right|_{H^{m,N}_\omega(I)}\|U\|_{L^2_\omega(I)} \quad (4.19)$$

Therefore the equation (4.17) can be written as follows:

$$\|I_N(J_1)\|_{L^2_\omega(I)} \le \sum_{i=0}^{N} CN^{-m}\max_{x\in I}\left|K(x,s(x,.))\right|_{H^{m,N}_\omega(I)}\|U\|_{L^2_\omega(I)}\,|F_i(x)|$$

Then

$$\|I_N(J_1)\|_{L^2_\omega(I)} \le CN^{-m}\max_{x\in I}\left|K(x,s(x,.))\right|_{H^{m,N}_\omega(I)}\|U\|_{L^2_\omega(I)}\sum_{i=0}^{N}|F_i(x)| - F(x,\alpha(x))\frac{d\alpha(x)}{dx}$$

Then the equation (4.27) can be written as:

$$\|J_3(x)\|_{L^2_\omega(I)} \le CN^{-1}\left\|K(x,x)e(x)+\int_{-1}^{x}K_x(x,s)e(s)ds\right\|_{L^2_\omega(I)}$$
$$\le CN^{-1}\|e\|_{L^2_\omega(I)} . \quad (4.28)$$

Then by using the equations (4.22), (4.24) and (4.28), we express the equation (4.16) as follows:

$$\|e(x)\|_{L^2_\omega(I)} \le CN^{-(2+m)}\max_{x\in I}\left|K(x,s(x,.))\right|_{H^{m,N}_\omega(I)}\left(\|e(x)\|_{L^2_\omega(I)} + \|u(x)\|_{L^2_\omega(I)}\right) + CN^{-m}|u|_{H^{m,N}_\omega(I)} + CN^{-1}\|e\|_{L^2_\omega(I)} . \quad (4.29)$$

Then

$$\|e(x)\|_{L^2_\omega(I)} \le CN^{-(2+m)}\max_{x\in I}\left|K(x,s(x,.))\right|_{H^{m,N}_\omega(I)}\|e(x)\|_{L^2_\omega(I)} + CN^{-(2+m)}\max_{x\in I}\left|K(x,s(x,.))\right|_{H^{m,N}_\omega(I)}\|u(x)\|_{L^2_\omega(I)} + CN^{-m}|u|_{H^{m,N}_\omega(I)} + CN^{-1}\|e\|_{L^2_\omega(I)}$$

If N be large enough, then we have:

$$\|e(x)\|_{L^2_\omega(I)} \le CN^{-(2+m)}\max_{x\in I}\left|K(x,s(x,.))\right|_{H^{m,N}_\omega(I)}\|u(x)\|_{L^2_\omega(I)} + CN^{-m}|u|_{H^{m,N}_\omega(I)}$$

4.2. Error analysis in L^∞ space

Theorem 4.2. Let u be the exact solution of the Volterra equation (2.1) and U be defined by (4.1) . If $u\in H^m_\omega(I)$ and $e(x)=U(x)-u(x)$, then for $m\ge 1$,

$$\|e(x)\|_{L^\infty_\omega(I)} \le CN^{-(2+m)}\max_{x\in I}\left|K(x,s(x,.))\right|_{H^{m,N}_\omega(I)}\|u\|_{L^2_\omega(I)} + CN^{\frac{1}{2}-m}|u|_{H^{m,N}_\omega(I)}$$

provided that N is sufficiently large, and C is a constant independent of N.

Proof:

Similar to the proof of Theorem 4.1, we get:

$$e(x) \le \int_{-1}^{x} e(s)ds + \left(I_N(J_1)+J_2(x)+J_3(x)\right) \quad (4.30)$$

By using the Lemma (3.4) we have:

$$\|e(x)\|_{L^\infty_\omega(I)} \le C\|I_N(J_1)+J_2(x)+J_3(x)\|_{L^\infty_\omega(I)}$$
$$\le C\left(\|I_N(J_1)\|_{L^\infty_\omega(I)}+\|J_2(x)\|_{L^\infty_\omega(I)}+\|J_3(x)\|_{L^\infty_\omega(I)}\right) \quad (4.31)$$

By using the equation (3.4) is expressed:

$$I_N(J_1)=\sum_{i=0}^{N}J_1(x_i)F_i(x)$$

By using the L^∞ norm can be written:

$$\|I_N(J_1)\|_{L^\infty_\omega(I)} = \left\|\sum_{i=0}^{N}J_1(x_i)F_i(x)\right\|_{L^\infty_\omega(I)} \le \sum_{i=0}^{N}|J_1(x_i)|\,|F_i(x)|. \quad (4.32)$$

By using the equation (4.19) we are obtained:

$$\|I_N(J_1)\|_{L^\infty_\omega(I)} \le \sum_{i=0}^{N} CN^{-m}\max_{x\in I}\left|K(x,s(x,.))\right|_{H^{m,N}_\omega(I)}\|U\|_{L^2_\omega(I)}\,|F_i(x)|.$$

Then

$$\|I_N(J_1)\|_{L^\infty_\omega(I)} \le CN^{-m}\max_{x\in I}\left|K(x,s(x,.))\right|_{H^{m,N}_\omega(I)}\|U\|_{L^2_\omega(I)}\sum_{i=0}^{N}|F_i(x)| + CN^{\frac{1}{2}-m}|u|_{H^{m,N}_\omega(I)} + CN^{-\frac{1}{2}}\|e\|_{L^\infty_\omega(I)} \quad (4.41)$$

Then

$$\|e(x)\|_{L^\infty_\omega(I)} \le CN^{-(2+m)}\max_{x\in I}\left|K(x,s(x,.))\right|_{H^{m,N}_\omega(I)}\|e(x)\|_{L^\infty_\omega(I)} + CN^{-(2+m)}\max_{x\in I}\left|K(x,s(x,.))\right|_{H^{m,N}_\omega(I)}\|u(x)\|_{L^2_\omega(I)} + CN^{\frac{1}{2}-m}|u|_{H^{m,N}_\omega(I)} + CN^{-\frac{1}{2}}\|e\|_{L^2_\omega(I)}.$$

If N be large enough, then we have:

$$\|e(x)\|_{L^2_\omega(I)} \le CN^{-(2+m)}\max_{x\in I}\left|K(x,s(x,.))\right|_{H^{m,N}_\omega(I)}\|u(x)\|_{L^2_\omega(I)} + CN^{\frac{1}{2}-m}|u|_{H^{m,N}_\omega(I)}.$$

V. Numerical Examples

Example 5.1. Consider the Volterra equation (2.1) with

$$K(x,s)=e^{xs}$$

$$g(x) = e^{4x} + \frac{1}{x+4}\left(e^{x(x+4)} - e^{-(x+4)}\right)$$

The corresponding exact solution is $u(x) = e^{4x}$.

This problem was solved in [2]. We use the Chebyshev Gauss points as the collocation points, therefore the points and their corresponding weights are as follows:

$$x_j = \cos\frac{(2j+1)\pi}{2N+2} \qquad , j = 0,1,\dots,N$$

$$\omega_j = \frac{\pi}{N+1} \qquad , j = 0,1,\dots,N$$

By using the numerical scheme (2.13), we will get numerical errors get for several values of N(table5.1-5.3).

TABLE 5.1

x	**The error estimate For N=4**		
	Approximate Solution	Exact Answer	Error
-0.9511	0.0021	0.0223	0.0202
-0.5878	0.2085	0.0953	0.1132
0	0.8466	1	0.1534
0.5878	10.5630	10.4982	0.0648
0.9511	42.9918	44.8983	1.9065

TABLE 5.2

x	**The error estimate For N=6**		
	Approximate Solution	Exact Answer	Error
-0.9749	0.0202	0.0203	0.0001
-0.7818	0.0486	0.0438	0.0048
-0.4339	0.1676	0.1763	0.0087
0	1.0035	1	0.0035
0.4339	5.6263	5.6723	0.0460
0.7818	22.5621	22.8100	0.2479
0.9749	48.5311	49.3827	0.8516

Also, according to theorem 4.1 can be obtained the error using the following equation

$$\|e(x)\|_{L^2_\omega(I)} \le CN^{-(2+m)} \max_{x\in I}\left|K(x,s(x,.))\right|_{H^{m,N}_\omega(I)} \|u\|_{L^2_\omega(I)} + CN^{-m}|u|_{H^{m,N}_\omega(I)}$$

That for several values of N, and by inserting $C = 10^{-4}$ in the above formula, the following table is derived

TABLE 5.3

	Example 5.1: The point error.				
m **N** **error**	6 6 3.2828e-004	8 8 1.4548e-005	10 10 3.9136e-007	12 12 7.0821e-009	14 14 2.2771e-011
m **N** **error**	16 16 9.3767e-013	18 18 7.8881e-015	20 20 6.1956e-017	22 22 5.2652e-019	24 24 5.2110e-021

Example 5.2. Consider the Volterra equation (2.1) with

$$K(x,s) = x\sin(s)$$

$$g(x) = \cos(x) + \frac{x}{4}(\cos(2) - \cos(2x))$$

The corresponding exact solution is $u(x) = \cos(x)$.

TABLE 5.4

	Example 5.2: The point error.				
m **N** **error**	6 6 5.5942e-008	8 8 2.4099e-010	10 10 5.6922e-013	12 12 8.4479e-016	14 14 8.5901e-019
m **N** **error**	16 16 6.3522e-022	18 18 3.5679e-025	20 20 1.5734e-028	22 22 5.5913e-032	24 24 1.6351e-035

RESULT

In this paper we used the Chebyshev-collocation spectral method to solve Volterra integral equations. By using this method the errors decay exponentially and the approximate answers are achieved with higher accuracy degree relative to Legendre-collocation spectral method. (The Legendre-collocation spectral method is the only spectral method that ever its convergence analysis is expressed)

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