Linear Algebra from Students' Perspectives

Ahmet ISIK*

Atatürk University

Fatih BAS**

Erzincan University

Muzaffer OKUR***

Erzincan University

Mehmet BEKDEMIR****

Erzincan University

Alper CİLTAS*****

Atatürk University

Abstract

This study aims to evaluate linear algebra course from students' perspectives. Within this scope, students' difficulties related to the linear algebra course, benefits of the course, the most and the least interesting topics in the course and suggestions for effective linear algebra instruction were considered in the study. Qualitative study conducted with 72 students enrolled in the primary mathematics teaching departments showed that students lived different problems in linear algebra; the course had positive effects on student's different skills; the most and the least interesting topics were matrix and vector spaces. Some suggestions were given about effective linear algebra teaching.

Key words

Linear algebra, students' perspectives, difficulties in linear algebra, benefits of linear algebra

*Prof. Dr., Department of Primary Mathematics Teaching, Atatürk University, Erzurum 25240, Turkey isik@atauni..edu.tr
** Assist. Prof. Dr., Department of Primary Mathematics Teaching, Erzincan University, Erzincan 24100, Turkey fbas@erzincan.edu.tr
*** Assoc. Prof. Dr., Primary Mathematics Teaching, Erzincan University, Erzincan 24100, Turkey mokur@erzincan.edu.tr
****Assoc. Prof. Dr., Primary Mathematics Teaching, Erzincan University, Erzincan 24100, Turkey fbas@erzincan.edu.tr
*****Assist. Prof. Dr., Department of Primary Mathematics Teaching, Atatürk University, Erzurum 25240, Turkey alperciltas@atauni.edu.tr

Introduction

Although there are methods about solutions of linear equation systems in Antique China period, the history of linear algebra is commonly accepted as being started with the work entitled as La Geometrie written in 1637 by Rene Descrates, and this belief still exists today (Haddad, 1999). The course which exists with the Analysis course within the first two years of sciences based departments' curricula of universities' some faculties (Dorier, 2002), is a branch of modern algebra questioning abstract systems which were created with the solutions of linear equation systems and were named as vector spaces (Konyalıoğlu, İpek and Işık, 2002).

Although the studies at university level mainly on determination of students' learning difficulties, operational problems in education system and how to overcome these barriers were initially about the Analysis course (Artigue, 1999), the studies begin to focus on teaching linear algebra in the last two decades before the year, 2002 (Dorier, 2002). The relevant studies according to their aims can be group as follow:

- Historical analyses showing some reasons of learners' learning difficulties and develop new curricula.
- Cognitive flexibility researches aiming to balance the use of geometry in linear algebra and conducted about linear algebra's formal structure.
- Researches aiming to evaluate linear algebra teaching performed with soft-wares (Aydın, 2009).

This research report mentions the studies containing students' difficulties in linear algebra course's teaching process and investigating the different aspects of the course. The first analyses of the linear algebra course from the historical and the epistemological views were done by Robinet (1986, cited in Dorier and Sierpinska, 2001). Studies in this way were continued with the studies of Dorier (1995a, 1995b). The common emphasis on the description of this course is linear algebra's being a difficult course in the eyes of students (Possani, Trigueros, Preciado and Lozano, 2010). In this concept, Dorier (2000)'s statement that students feel themselves in a different universe when they have to learn many new concepts and lost the connection between their current and prior knowledge, becomes important for this situation.

The answer of why linear algebra is accepted as a difficult course by most of the learners is primarily hidden in the purpose of this course and its historical development. In this concept, the linear algebra course is different from the Analysis course existing in the first two years of sciences based departments' curricula of universities' some faculties. Because the Analysis was arisen from the need of answering a number of questions such as how to find slope of the tangent which is drawn on a catenary (1), calculate the ratio of exchange (2), calculate the area under a catenary (3), and calculate momentum when acceleration is given from this perspective (4) (Haddad, 1999). On the other hand, the concepts like vector spaces stated in Konyalıoğlu et al. (2002)'s the definition of linear algebra, have a purpose to solve different problem situations with the same agent, not a function of finding a solution to the new problem (Dorier, 1995b).

As a result of this situation, linear algebra is dealt with abstract constructs showing systems having different features and various concepts (Harel, 1989). Then, linear algebra's being abstract can be accepted as another reason of the students' living difficulty in the algebra course. Harel (1989) emphasized why it is unsurprising for high school students and freshmen students could not solve

the problems by using abstract constructs and principles by stating high school curricula's being inefficient to support appropriate knowledge to students about abstract constructs and how to transfer the certain solution do problems into different contexts. According to Dubinsky (1997), besides students' lack of information about necessary concept for teaching these concepts, teachers' not giving opportunity to students for constructing their own ideas about these concepts lies in the basis of students' living difficulties in learning linear algebra's topics such as space, base, linear independency, linear transformation and matrixes.

According to Aydın (2007), the reason of this problem in Turkey is the lack of questions in university entrance examinations held for students after high school graduation to enter university. In addition this fundamental deficiency, another problem is about the questions in these examinations (İşleyen and Işık, 2005). Because students place answering correctly to the questions in the shortest time constraint to the forefront in these examinations, and as a result they learn only operational knowledge skills not problem-solving and reasoning skills (İşleyen and Işık, 2005). Students' expecting the course teacher's answering to the questions asked for their solving for the intra-class activity in the linear algebra course is also another reason of the difficulties and cautions approaches to linear algebra concepts.

Another reason of students' living difficult in the linear algebra course is students' being unaware of linear algebra's usage area (Harel, 1989) in spite of linear algebra's importance for different areas and its being compulsory in many disciplines' educational curricula (Poassani et al., 2010).

The diversity of presentation styles and semiotic form of representations in linear algebra can be addressed as other reasons of why learners live problems in learning linear algebra (Aydın, 2009). Hillel (2000) defines the languages used in linear algebra:

- Abstract form including such the concepts: vector space, subspace, dimension, kernel set.
- Algebraic form including the concepts: matrix, rank, solutions of equation systems.
- Geometric form containing the concepts point, line, plane, geometric transformation.

Students should be able to do transitions between different forms of the concepts which they would learn, as well as discriminate different forms of the representations (Aydın, 2009). But it can be said that students live problems in discriminating the concepts from each other and doing transitions between the representation of the concepts (Duval, 2000 cited in Aydın, 2009). The reasons of students' living problems were evaluated under three headings by Haddad (1999) as below;

- Linear algebra's structure,
- Linear algebra's teaching,
- How students learn and deal with linear algebra

These headings which are the resources of students' attitude towards linear algebra are selected as baseline in this study. However, when the application area of linear algebra and its place in the curricula of education faculties' science based departments are thought, it becomes more important to determine lived difficulties and analyze the course from the viewpoints of learners. When considered that linear algebra's necessity as teaching, and its existence in both mathematics and different areas of life (Işık, Kaplan, Işık, Konyalıoğlu, Güler and Kar, 2011), it is believed that the results gained at the end of the study would be helpful in linear algebra course's planning process to decrease the difficulties experienced by the student.

Aim of the Study

The aim of this study is to evaluate the linear algebra course from the viewpoints of university students. The study addresses the following research questions;

1. What are the difficulties experienced by the students in Linear Algebra course?
2. What are the benefits of Linear Algebra course to students?
3. What are the least and the most interesting topics in Linear Algebra course for the students?
4. What are the suggestions of the students for a better Linear Algebra course teaching?

Methodology

Case study method from qualitative research approaches was adopted in this study In case studies, one or more events, contexts, programs, social groups or other related systems are analyzed in depth (McMillan, 2000).

Subjects

The study was conducted in 2011 with 72 students being educated in the primary mathematics teaching departments of two universities' faculties of education which are located in eastern part of Turkey. The selection process of sampling group was done in two ways by being used typical case and maximum variation sampling methods from purposive sampling techniques.

The reasons of conducting study in two universities are to lessen the effects deriving from the instructors and to reach a broader perspective about the course. Transportation facilities to these universities became another important factor considered by the researchers.

Data Collection Instrument and Data Collection Process

The data were collected in focus group discussions with a semi-structured interview form prepared by being gathered ideas of two experts in linear algebra, education area and qualitative research methods. The semi-structured interview form included following questions: 1) What are your negative thoughts about Linear Algebra course? 2) What are the benefits of this course to you? 3) What are the most interesting topics for you in this course?,4) What are the least interesting topics for you in this course? 5) What can be done for more effective teaching of Linear Algebra course?. Besides these questions, some other questions to get information about the subjects and ease the understanding of the subjects' answers to questions were asked to the subjects during the interview. The data were collected in 8 sessions each of which lasted about 35 minutes. Interviews were recorded both camera and voice record device in order to avoid probable data loss. During the interviews, the data gathered by the researchers were restated to the interviewees to verify the collected data. After the interviews, the records were transcribed and turned into written texts.

Data Analysis

The data related to "What are the difficulties experienced by the students in Linear Algebra course?" were analyzed descriptively. Descriptive analysis is a method used in the studies containing pre-

determined conceptual structures (Yıldırım and Şimşek, 2008). In the analysis process of students' difficulties in linear algebra course, the themes including the structure of linear algebra, teaching linear algebra, and students' learning process of linear algebra which are constructed by Haddad (1999) were used as baseline. As the study focuses on the evaluation of the linear algebra course from students' perspectives, the difficulties deriving from the course instructors which are partially related to teaching linear algebra in this conceptual structure were not considered in the study. In addition, the subjects were requested not to emphasize the difficulties deriving from the course instructors during the interviews. Respectively, the data related to the second, third and fourth research questions: "What are the benefits of Linear Algebra course to students?, What are the least and the most interesting concepts in in Linear Algebra course for the students?, What are the suggestions of the students for a better Linear Algebra course teaching?" were evaluated with content analysis aiming at reach the concepts and the relations about the collected data (Yıldırım and Şimşek, 2008).

In the analysis process, firstly the students' answers to the questions in the interviews were analyzed and irrelevant answers about these questions were disregarded. Then, the codes comprising of all data and consisted with the conceptual structure related to the first research question and other research questions were determined. As a last, analyzed data were combined under the appropriate categories about them. In order to increase coding reliability, the used categories, the codes and the excerpts were evaluated by an expert good at qualitative research methods. During the presentation of the students' excepts, the subjects were written as "S-1, S-2, S-3, … S72".

Findings

In this section, the findings about the first research question *"What are the difficulties experienced by the students in Linear Algebra course?"* were presented in tables by being grouped under headings "reasons about course structure, reasons about course teaching and reasons about students".

The reasons of the difficulties in the linear algebra course stemming from course structure according to the viewpoints of the students are presented in Table 1.

Table 1. The difficulties in linear algebra course stemming from course structure

Reasons about Course Structure	frequency(f)
Abstractness of the topics	21
Any relations between the topics	13
Not transferring the learned knowledge to the future	12
Complexity of the topics	8
Not knowing the usage area of the topics	6
Extensive symbol usage and complexity in the symbols	2
Topic broadness	2

High numbers of definition and theorems	1

As seen in the Table 1, reasons stemming from the course structure were subsumed under 8 categories. Students' statements about the first four main reasons "abstractness of the topics (21), any relations between the topics (13), not transferring the learned knowledge to the future (12), complexity of the topics (8)" as follow:

The student, S-29's view on the abstractness of the topics is as follows:

S-29: "From my point of view, linear algebra course requires much practice; its reason is the course's being completely a theoretical course. Because of the abstractness of the topics, understating this course becomes really difficult. In sum, I wish this course never existed."

The subject, S-39, emphasizing the importance of setting relations between the topic stated that:

S-39 "As the topics in linear algebra are generally related to each other, when you do not know the first topic, you will live difficulty in understanding next topics. When we miss a topic, later the topic becomes impossible to comprehend for us."

S-59 expressing that they will not transfer the learned knowledge to the future, said on this issue:

S-59: "… As the students of primary mathematics teaching department, we believe that we will not use linear algebra in the future."

S-24 said that the topics in this course are abstract and went on by saying like:

S-24: "… We still have fears, doubts about the course. Because this course is not an easy course."

As seen in the Table 1, the other reasons about the course structure are the complexity of the topics, not knowing the usage area of the topics, extensive symbol usage and complexity in the symbols in the course, topic broadness and high numbers of definition and theorems.

The reasons of the difficulties in the linear algebra course stemming from course teaching according to the viewpoints of the students are presented in Table 2.

Table 2. Difficulties in linear algebra course stemming from course teaching

Reasons about Course Teaching	frequency(f)
Education system	14
Lack of additional resources	6
Insufficient course hours	4

14 of the subjects linked the problems arising from course teaching to the education system; 6 subjects said they live problems about reaching additional course resources and 4 subjects emphasized the insufficient course hours. Some of the subjects' views on these issues as follow:

S-3: "In my opinion, the teachers' guidance such as this topic is important or not, or study this or do not study that in university entrance examination system which allows us to enter university, poses problems in learning for us. As those teachers only taught the easiest way of solving a question, not teaching the correct

way of solving, linear algebra course does not fit with this kind of education. Because this course is mostly about comprehension…"

S-43: "… In fact, one of the reasons of our living difficulties in this course is also our lack of material about linear algebra. … As there is a shortage about the resources, we could not find different course resources."

S-26: "… Limited time makes us directly pass some of the concepts in topics, in other words definitions. If this course is taught during sufficient course hours, maybe these topics can be taught well and logic of the topic can be understood."

The reasons of the difficulties in the linear algebra course stemming from students according to the viewpoints of the students are presented in Table 3.

Table 3. Difficulties in linear algebra course stemming from students

Reasons about Students	frequency(f)
Lack of preparedness	29
Not internalizing of the topics	29
Prejudice about the course	7
The thought of only passing the course	4
Not studying the topics	3
Not asking questions	2
Fear of examination	1

According to the Table 3, reasons abut students were subsumed under 7 categories. It is seen that most of the subjects emphasized the reasons as follows: lack of preparedness (29), not internalizing of the topics (29) and prejudice about the course (7). The views of the subjects, respectively S-11, S-60 and S-23 can be given as an example to these aspects.

S-11: "In other courses, at least we have some preparedness coming from our high school years. But linear algebra is totally a new course for us and we study it for the first time."

S-60: "… We were not trying to understand how to solve or how to reach conclusion, we were just trying to memorize it. We were only trying to memorize like this question is solved in this way. Then, when we forgot even very small part of it, we lost everything in the examination. In other words, we were not internalizing how to solve the questions."

S-23: "We often studied linear algebra in a prejudiced way. Therefore, we could not get a clear base for this course."

In addition, the students expressed other reasons deriving from themselves such as their thoughts of only passing the course, not studying the topics enough, not asking questions, fear of examination as seen in the Table 3.

The findings about the second research question "What are the benefits of Linear Algebra course to students?" are presented in Table 4.

Table 4. Benefits of linear algebra course to students

Educational Attainments	frequency(f)
Develop mathematical and abstract thinking skills	9
Bring effective study skills	5
Develop the use of mathematical language skills	4
Increase conceptual and operational knowledge level	3

As presented in the Table 4, 9 subjects emphasized that the linear algebra course developed students' mathematical and abstract thinking skills. Having this view, S-42 stated that:

S-42: "… In the course, newly met theorems were presented to us and according to these new theorems we develop new things in our minds. With this way, I think our abstract thinking skills were developed."

Another point of the benefits for the students is bringing effective study skills stated by 5 subjects. On this issue, S-71 said:

S-71: "… As this course requires daily study plan, in other words it brought us systematic study skills. That is daily systematic study discipline."

In addition, the subjects expressed that linear algebra course developed the use of mathematical language skills and increased their conceptual and practical knowledge levels. Respectively students' views on these sentences are as follow:

S-68"… With this course, we learned how to use mathematical language and developed in some degree. As we are mathematicians, we should know and use mathematical knowledge."

S-27"… Our conceptual knowledge was developed. … As linear algebra based on the concepts, we made operations by learning these concepts. Then our operational knowledge was developed, too."

With the third research question *"What are the least and the most interesting topics in Linear Algebra course for the students?"*, it is aimed to determine students' views about the least and the most interesting topics in the course. The findings related to the third question are presented in Table 5.

Table 5. Interesting and uninteresting topics in the content of linear algebra course

Interesting Topics	frequency(f)	Uninteresting Topics	frequency(f)
Matrix	13	Vector Spaces	14
Elementary Operations	9	Sub- Vector Spaces	8
Determinant	8	Linear Algebra II Topics	8
Linear Transform	7	None of the topics	3
First Term Topics	4	Determinant	2
Homogenous System of	2	Matrix	1

Linear Equations			
Vector Spaces	2	Base and Size	1
Converting Bases to Each Other	1	Field	1
Basis and Size	1	Questions Based on Theorem	1
Linear Dependence and Independence	1	All topics	1
First Topics	1		

The reasons of students' categorizing topics in this way are mainly about the situation of the students' having prior knowledge about the topics or not, and the topics' having arithmetical operations or not. In this concept, the subject, S-58 saying that he is interested in matrixes, gave reasons about this as below:

S-58: "… because there were numerical data and we learned them before, it looked like a puzzle. That is, it was enjoyable for us."

Another subject, S-66 stating that he is uninterested in vector spaces expressed this situation by reasoning like:

S-66: "… What is its reason? Abstract algebra knowledge was necessary for the course. Such as abelian group, commutative property, closure property. I learned them from abstract mathematic course book. I partially understood them but I could not fully internalize them. Though I know vector spaces topics, I did not like the topic."

The findings related to the fourth research question *"What are the suggestions of the students for a better Linear Algebra course teaching?"* are presented in Table 6.

Table 6. Suggestions for effective linear algebra teaching

Suggestions	frequency(f)
Topics can be concretized.	21
Practicing can be increased.	16
Topics can be taken in a much detailed way.	7
Much emphasis can be given on students' preparedness.	3
Much effort can be given to develop positive attitude.	1
Easier language can be adopted.	1

As seen in the Table 6, the students' suggestions for a better linear algebra course teaching were subsumed under 6 categories.

S-25 believing in the idea that being concretized of the topics would be useful, gave rationale by saying:

S-25: "… before creating prejudice to students, teacher should show where to use these knowledge in the first course. It is important to attract students' attention."

S-40 accepting more practice would be helpful stated his thought on this issue with this except:

S-40: "… Questions are solved in the class, but it is limited because of the course hour. In my point of view, more solved questions in algebra means much long lasting knowledge. For instance, ten questions can be given for each topic. Then, we try to solve them in one or two days. As a last teacher help us to check them in an allocated time."

S-8 believing that teaching topics in a more detailed way would be effective expressed his thought by saying:

S-8: "… When a formula is presented, only a part of it is taught. So, that formula is not totally covered. No rationale is given about the formula. Maybe we can understand topics in linear algebra I-II, but we cannot solve the problems when we see intermediate level knowledge. We cannot go on solution with that formula. As we do not know much and the rationale of the formulas…"

S-24 offering more emphasis on prior knowledge, said:

S-24: "… For the preparedness, linear algebra topics can be taught under a course in high school, even if it does not become a separate course named linear algebra."

S-11 suggesting easier language use in the course said:

S-11: "I think the use of easy language is also important in the simplification process of the course."

Results and Discussions

The reasons of the difficulties related to the linear algebra course by being looked from students' perspectives were evaluated within three headings such as course structure, course teaching and students.

While emphasizing that abstractness and complexity of the topics constitutes the difficulties deriving from the structure of the course, most of the subjects said they lived problems in conducting relations between the topics. These results are in parallel to the studies, Poassani et al. (2010) reporting that linear algebra course is a difficult course from students' perspectives, and Dorier (2000) reporting that students live difficulty in relations between the concepts. Another study which has similar study is Duval (2000) which reports that students live problems in learning algebra because of the symbolic representations of the concepts. In addition to these results, students' negative beliefs on transferring the learned knowledge to the future and having little knowledge about the usage area of the topics are among the reasons deriving from the course structure. This result is in parallel to the results of Harel (1989) which underlines that students are unfamiliar with the usage are of linear algebra. Other reasons stated by the students about the course structure are as follow: Extensive symbol usage and complexity in the symbols, topic broadness and high numbers of definition and theorems in course content.

The subjects emphasized the problems based on the education system such as little exposure to the concepts included in the linear algebra course in their formal education processes and university

entrance examinations' lack of these concepts. The same results were indicated by Aydın (2007), and it is thought that these kinds of problems can be lessened by including the topics related to the linear algebra course. In addition, some student stated that they have insufficient numbers of additional resources and less course hours.

Difficulties lived in the linear algebra course deriving from the students according to subjects can be ordered as students' lack of prior knowledge, not internalizing of the topics, prejudice about the course, thoughts of only passing the course, not studying the topics, not asking questions and their fear towards the examinations.

Students' views about the educational attainments of the linear algebra course are categorized as follow: mathematical and abstract thinking skills, effective study skills, the use of mathematical language skills and developing conceptual and operational knowledge skills.

While students' most interested topic are matrix and elementary operations which contain operational knowledge and prior knowledge, their least interested topics are vector spaces and sub-vector spaces which are abstract topics and students do not have prior knowledge.

As suggested in Aydın's study 2007, this longitudinal study has revealed that topics can be concretized, practicing linear algebra can be increased, topics can be taken in a much detailed way, students' prior knowledge can be considered before teaching, an easier language can be used in the class and positive attitude towards the course can be developed for more effective algebra teaching.

References

Artigue, M. (1999). The teaching and learning of mathematics at the university level: crucial questions for contemporary research in education. *Notices of The American Matematical Society, 46*(11), 1377-1385.

Aydın, S. (2007). Bazı özel öğretim yöntemlerinin lineer cebir öğrenimine etkileri. *Elektronik Sosyal Bilimler Dergisi, 6*(19), 214-223.

Aydın, S. (2009). Lineer cebri öğretimi üzerine. *İnönü Üniversitesi Eğitim Fakültesi Dergisi, 10*(1), 93-105.

Dorier, J. L. (1995a). A general outline of the genesis of vector space theory. *Historia Mathematica, 22*(3), 227-261.

Dorier, J. L. (1995b). Meta level in the teaching of unifying and generalizing concepts in mathematics. *Educational Studies in Mathematics, 29*(2), 175-197.

Dorier, J. L. (2002). Teaching Linear Algebra at University. *In the Proceedings of the International Congress of Mathematicians,3*, 875-884.

Dorier, J. L., & Sierpinska, A. (2002). Research into the teaching and learning of linear algebra, *The Teaching and Learning of Mathematics at University Level New ICMI Study Series,7*, 255-273.

Dubinsky, E. (1997). *Some Thoughts on a first course ın linear algebra at the college level*. Retrieved September 26, 1995, from http://www.math.kent.edu/~edd/LinearAlgebra.pdf

Haddad, M. (1999). *Difficulties in the learning an teachin of linear algebra- a personal experience.* Unpublished master's thesis, Concordia University.

Harel, G. (1989). Learning and teaching linear algebra: difficulties and an alternative approach to visualizing concepts and processes. *Focus on Learning Problems in Mathematics, 11*(2), 139-148.

Hillel, J. (2000). Modes of description and the problem repsenetation in linear algebra. In J.L. Dorierv(E.d). *On The Teaching of Linear Algebra* (191-207). Dordrecht: Kluwer Academic Publishers.

Işık, A., Kaplan, A., Işık, C., Konyalıoğlu, A. C., Güler, G., & Kar, T. (2011). Problem based learning in linear algebra, *International Journal of Humanities and Social Science, 1*(10), 187-196.

İşleyen, T., & Işık, A. (2005). Alt vektör uzayı kavramının kavramsal öğrenilmesi üzerine. *Kazım Karabekir Eğitim Fakültesi, 11*, 492-501.

Konyalıoğlu, A. C., İpek, A. S., & Işık, A. (2002). Farklı bir yaklaşımla vektör uzayı kavramının öğretimi. *Kastamonu Eğitim Dergisi, 10*(2), 325-332.

McMillan, J. H. (2000). *Educational research fundamentals for the consumer.* USA: Longman.

Possani, E., Trigueros, M, Preciado, J. G., & Lozano M. D. (2010). Use of models in the teaching of linear algebra. *Linear Algebra and its Applications, 432*, 2125-2140.

Yıldırım, A., & Şimşek, H. (2008). *Sosyal Bilimlerde Nitel Araştırma Yöntemleri* (7. Baskı). Ankara: Seçkin Yayıncılık.