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CALCULATED OF REGULARIZED TRACE OF A EVEN ORDER DIFFERENTIAL EQUATION IN FINITE INTERVAL

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ABSTRACT

In this study ,the regularized trace formula of the even ordered differential equation with two terms given on a finite interval is examined
Keywords: Eigenvalue, eigenfunction, asymptotic formula, unitar matrix.

SONLU ARALIKTA ÇİFT MERTEBEDEN BİR DİFERANSİYEL DENKLEMİNİN DÜZENLİ İZİNİN HESAPLANMASI

ÖZET

Bu çalışmada sonlu aralıkta verilmiş çift mertebeden iki terimli bir diferansiyel denklemin düzenli iz formülü incelenmiştir.
Anahtar Sözcükler: Özdeğer, özfonksiyon, asimtotik ifade, unitar matris.

1. INTRODUCTION

As known, trace of n dimensional matrix is the sum of n eigenvalues. The eigenvalues of Sturm-Liouville problem satisfy the following condition

$$\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$$

and

$$\lim_{n\to\infty} \lambda_n = \infty$$

From here, it is seen that it hasn't finite limit of sum of numbers $\left\{\lambda_k\right\}_{k=1}^{\infty}$ because of this there is no meaning to say about the trace of the Sturm-Liouville problem directly. Firstly; I.M. Gelfand and B.M. Levitan found the series sum which was formed by numbers $\left\{\lambda_k - k^2\right\}_{k=1}^{\infty}$.

The sum of the series was called regularized trace of the Sturm-Liouville problem. Regularized trace formula which was mentioned was found with more simple method by

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L.A. Dikii [12] Later, the articles [5] - [11] and the books [1] - [2] were written about this subject. We calculated the regularized trace of an even order differential equation with Dikii's method.

In this paper, the regularized trace formula for the operator L in $L_2(0,\pi)$ space which is defined with differential equation

$$(-1)^m y^{(2m)} + q(x)y = \lambda y \qquad , \qquad 0 < x < \pi \tag{1}$$

and with boundary conditions

$$y(0) = y^{''}(0) = ... = y^{(2m-2)}(0) = y(\pi) = y^{''}(\pi).......... = y^{(2m-2)}(\pi) = 0 \tag{2}$$

is calculated

Here, $q(x)$ is a real valued , continuous function in $[0,\pi]$. Initially , the regularized trace formula of Sturm-Liouville operator was calculated by I.M.Gelfand and B.M.Levitan [1]. Later, this study was generalized for different operators. Excessive informations and references that are given in [2],[4] and [5]-[11] ,can be shown as recent studies. Trace formula obtained in [1], was proved by an algebric method by L.A.Dikii [12].

We obtained the regularized trace formula of (1)-(2) by Dikii method which was found before in [4]. While $q(x)$ =0,it is clear that

$$\mu_n = n^{2m} \qquad (n = 1,2,.....)$$

are the eigenvalues of operator L and

$$\psi_n(x) = \sqrt{\frac{2}{\pi}}\sin nx$$

are the orthonormal eigenfunctions corresponding to this eigenvalues .

As known from [4](see also[13]) the eigenvalues and the orthonormal eigenfunctions of L have following asymptotic expressions.

$$\lambda_n = n^{2m} + \frac{1}{\pi}\int_0^{\pi} q(x)\,dx + O\left(\frac{1}{n^2}\right)$$

$$\varphi(x) = \psi_n(x) + O\left(\frac{1}{n}\right)$$

The results which were obtained are given at following theorem.

Theorem: If $\int_0^{\pi} q(x)dx = 0,$ then

$$\sum_{k=1}^{\infty}\left(k^{2m} - \lambda_k\right) = \frac{q(0)+q(\pi)}{4}$$

The series , at the left side of this equality are called reqularized trace of operator L.
Proof: To prove this theorem , it is sufficient to show that

$$\lim_{N\to\infty}\sum_{n=1}^{N}\left[(\varphi_n, L\varphi_n) - (\psi_n, L\psi_n)\right] = 0 \tag{3}$$

Really , by considering the following equations

$$\left(\varphi_n, L\varphi_n\right) = \lambda_n \quad , \quad (\psi_n, L\psi_n) = n^{2m} + (\psi_n, q\psi_n) .$$

It can be written as

$$\sum_{n=1}^{N}\left[\left(\psi_n, L\psi_n\right)-\left(\varphi_n, L\varphi_n\right)\right]=\sum_{n=1}^{N}\left(n^{2m}-\lambda_n\right)+\sum_{n=1}^{N}\left(\psi_n, q\psi_n\right).$$

Let we write $\sum_{n=1}^{N}(\psi_n, q\psi_n)$ summation as below

$$\sum_{n=1}^{N}(\psi_n, q\psi_n)=\frac{2}{\pi}\sum_{n=1}^{N}\int_0^{\pi} q(x)\sin^2 nxdx$$

$$=-\frac{1}{4}\sum_{n=2}^{2N}\sqrt{\frac{2}{\pi}}\cos(k.0)\int_0^{\pi} q(x)\sqrt{\frac{2}{\pi}}\cos kxdx$$

$$-\frac{1}{4}\sum_{n=1}^{2N}\sqrt{\frac{2}{\pi}}\cos k.\pi\int_0^{\pi} q(x)\sqrt{\frac{2}{\pi}}\cos kxdx$$

Here , if the expansion formula of $q(x)$ is considered according to cosinus in $[0,\pi]$,

$$\lim_{N\to\infty}\sum_{n=1}^{N}\left(\psi_n, q\psi_n\right)=-\frac{q(0)+q(\pi)}{4}$$

is seen.

Considering the study [12] ,we survey the transfer matrix from orthonormal basis $\{\varphi_i\}$ to orthonormal basis $\{\psi_j\}$. U_{ik} is a Unitar matrix. From the formula

$$\sum_{k=1}^{\infty}U_{ik}^2=\sum_{i=1}^{\infty}U_{ik}^2=1 \tag{4}$$

$$L\psi_k=k^{2m}\psi_k+q(x)\psi_k$$

it can be written as

$$\left(L\psi_k,\varphi_i\right)=k^{2m}\left(\psi_k,\varphi_i\right)+\left(q\psi_k,\varphi_i\right).$$

Here, if we consider that

$$\left(L\psi_k,\varphi_i\right)=\left(\psi_k, L\varphi_i\right)=\lambda_i\left(\psi_k,\varphi_i\right)$$

$$\lambda_i\left(\psi_k,\varphi_i\right)=k^{2m}\left(\psi_k,\varphi_i\right)+\left(q\psi_k,\varphi_i\right)$$ and

$$\left(\lambda_i-k^{2m}\right)\left(\psi_k,\varphi_i\right)=\left(q\psi_k,\varphi_i\right),$$

from the last expression it is found that

$$\sum_{i=1}^{\infty}\left(\lambda_i-k^{2m}\right)^2\left(\psi_k,\varphi_i\right)=\left\|q\psi_k\right\|^2<const$$

and

$$\sum_{i=1}^{\infty}\left(\lambda_i-n^{2m}\right)^2 U_{ik}<const \tag{5}$$

Because of $\int_0^{\pi} q(x)dx = 0$,the asymptotic formulas that we mentioned before, obtained as

$$\lambda_k = k^{2m} + O(\frac{1}{k^2}) \quad , \quad \varphi_k = \psi_k + O(\frac{1}{k})$$

From the equality (5), by doing similar operations for every natural number N while k$\leq N$, it is seen that

$$\sum_{i=N+1}^{\infty} (\lambda_i - \lambda_k) U_{ik}^2 < \frac{const}{(N+1)^{2m} - k^{2m}} \tag{6}$$

is true.

According to [12],if we consider

$$\sum_{k=1}^{N} (\psi_k, L\psi_k) = \sum_{k=1}^{N}\sum_{i=1}^{\infty} \lambda_i U_{ik}^2$$

and

$$\sum_{k=1}^{N} (\varphi_k, L\varphi_k) = \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} \lambda_k U_{ki}^2$$

it becomes that

$$\sum_{k=1}^{N} \left[(\psi_k, L\psi_k) - (\varphi_k, L\varphi_k) \right] = \sum_{k=1}^{N}\sum_{i=1}^{\infty} (\lambda_i - \lambda_k) U_{ik}^2 + \sum_{k=1}^{N}\sum_{i=N+1}^{\infty} \lambda_k \left(U_{ik}^2 - U_{ki}^2 \right) \tag{7}$$

According to (6)

$$\sum_{k=1}^{N}\sum_{i=N+1}^{\infty} (\lambda_i - \lambda_k) U_{ik}^2 < \sum_{k=1}^{N} \frac{const}{(N+1)^{2m} - k^{2m}}$$

While $N \to \infty$, the right side of above inequality is written as

$$\sum_{n=1}^{N} \frac{1}{(N+1)^{2m} - k^{2m}} \leq \frac{1}{(N+1)^{2m} - N^{2m}} + \int_1^N \frac{dx}{(N+1)^{2m} - x^{2m}}$$

$$= \frac{1}{\left((N+1)^m - N^m\right)\left((N+1)^m + N^m\right)} + \frac{N+1}{(N+1)^{2m}} \int_1^N \frac{d\frac{x}{N+1}}{1 - \left(\frac{x}{N+1}\right)^{2m}}$$

$$\leq \frac{1}{2N^m} + \frac{1}{(N+1)^{2m-1}} \int_{\frac{1}{N+1}}^{\frac{N}{N+1}} \frac{du}{1-u^{2m}} = \frac{1}{2N^m} + \frac{1}{(N+1)^{2m-1}} \int_{\frac{1}{N+1}}^{\frac{N}{N+1}} \frac{1}{2}\left(\frac{1}{1+u^m} + \frac{1}{1-u^m} \right) du$$

$$= \frac{1}{2N^{2m}} + \frac{1}{(N+1)^{2m-1}} \left(\frac{1}{2} \int_{\frac{1}{N+1}}^{\frac{N}{N+1}} \frac{du}{1 + \left(\frac{1}{N+1}\right)^m} + \frac{1}{2} \int_{\frac{1}{N+1}}^{\frac{N}{N+1}} \frac{du}{1-u^m} \right)$$

$$= \frac{1}{2N^{2m}} + \frac{1}{(N+1)^{2m-1}} \frac{1}{2} \left[\frac{1}{1+\frac{1}{(N+1)^m}} \left(\frac{N}{N+1} - \frac{1}{N+1} \right) + \int_{\frac{1}{N+1}}^{\frac{N}{N+1}} \frac{du}{1-u^m} \right]$$

$$\le \frac{1}{2N^{2m}} + \frac{const}{(N+1)^{2m-1}} + \frac{1}{2} \frac{1}{(N+1)^{2m-1}} \int_{\frac{1}{N+1}}^{\frac{N}{N+1}} \frac{du}{1-u^m} \sim \frac{const}{N^{2m-1}} +$$

$$+ \frac{1}{2} \frac{1}{(N+1)^{2m-1}} \int_{\frac{1}{N+1}}^{\frac{N}{N+1}} \frac{du}{1-u^m} \qquad (8)$$

for simplicity, by getting m=4, we restrict the last sum as:

$$\frac{1}{2} \frac{1}{(N+1)^{2m-1}} \int_{\frac{1}{N+1}}^{\frac{N}{N+1}} \frac{du}{1-u^4} = \frac{1}{2} \frac{1}{(N+1)^{2m-1}} \left(\frac{1}{2} \int_{\frac{1}{N+1}}^{\frac{N}{N+1}} \frac{du}{1+u^2} + \frac{1}{2} \int_{\frac{1}{N+1}}^{\frac{N}{N+1}} \frac{du}{1-u^2} \right)$$

$$= \frac{1}{2} \frac{1}{(N+1)^{2m-1}} \left(\frac{1}{2} \int_{\frac{1}{N+1}}^{\frac{N}{N+1}} \frac{1}{1+\frac{1}{(N+1)^2}} du + \frac{1}{2} \int_{\frac{1}{N+1}}^{\frac{N}{N+1}} \frac{du}{1-u^2} \right) \sim$$

$$\sim \frac{1}{4} \frac{1}{(N+1)^{2m-1}} \cdot \frac{1}{1+\frac{1}{(N+1)^2}} \left[\left(\frac{N}{N+1} - \frac{1}{N+1} \right) + \ell nN \right] \qquad (9)$$

Thus while m=4, from the (8) and (9), while $N \to \infty$,

$$\sum_{k=1}^{N} \sum_{i=N+1}^{\infty} (\lambda_i - \lambda_k) u_{ik}^2 \le \frac{const}{(N+1)^{2m-1}} \ell nN \to 0$$

is found.

Thus, while $N \to \infty$, it is showed that the first sum of the right side of (7) approachs to zero .The second sum in (7) is restricted as below.

If the below equation

$$u_{ik} + u_{ki} = -(\varphi_i - \psi_i, \varphi_k - \psi_k)$$

and the asymptotic expression of eigenfunctions of L is considered

$$|u_{ik} + u_{ki}| \le \|\varphi_i - \psi_i\| . \|\varphi_k - \psi_k\| \le \frac{const}{i.k}$$

is obtained.

If the similar operations are done according to [12],

$$\sum_{k=N+1}^{\infty}\left|u_{ik}^{2}-u_{ki}^{2}\right|<\frac{const}{k\sqrt{N+1}\left[\left(N+1\right)^{2m}-k^{2m}\right]}$$

is found.
Here, from the second sum of (7)

$$\sum_{k=1}^{N}\lambda_k\sum_{i=N+1}^{\infty}\left|u_{ik}^{2}-u_{ki}^{2}\right|<const\sum_{k=1}^{N}\frac{k^{2m}}{k\sqrt{N+1}\left[\left(N+1\right)^{2m}-k^{2m}\right]}$$

$$< const\left(N+1\right)^{2m-1-\frac{1}{2}}\sum_{k=1}^{N}\frac{1}{\left(N+1\right)^{2m}-k^{2m}}$$

$$\sim const\left(N+1\right)^{2m-1-\frac{1}{2}}\frac{\ell nN}{\left(N+1\right)^{2m-1}}\underset{N\to\infty}{\to}0$$

is obtained.
Thus, it satisfies the equality (3) and according this accuracy, the theorem is proved.
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