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SOME PROPERTIES CONCERNING THE HYPERSURFACES OF A WEYL SPACE

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ABSTRACT

Let W_n be a hypersurface of the Weyl space W_{n+1} . Let $\overset{*}{\underset{i}{v}}(i = 1, 2, ..., n)$ be tangent vector fields belonging to W_n and n be the normalized normal vector field of W_n. Consider the $(n+1)$ - net $\left(\overset{*}{\underset{1}{v}}, \overset{*}{\underset{2}{v}}, ..., \overset{*}{\underset{n}{v}}, n\right)$. By using the prolonged covariant differentiation, we first obtain the set of formulas corresponding to the Frenet formulas for W_n associated with a curve C on W_n having the tangent vector field $\underset{1}{v}$. We then, derive two invariants concerning the orthogonal ennuple $(\underset{1}{v}, \underset{2}{v}, ..., \underset{n}{v})$, $\underset{i}{v}(i = 1, 2, ..., n)$ being differentiable vector fields on W_n .

Keywords: Weyl space, Net of curves, Prolonged covariant derivative.
MSC number/numarası: 57R55, 58A99.

BİR WEYL UZAYININ GÖZÖNÜNE ALINAN HİPER YÜZEYLERİNİN BAZI ÖZELLİKLERİ

ÖZET

W_n , W_{n+1} Weyl uzayının bir hiperyüzeyi olsun. $\overset{*}{\underset{i}{v}}(i = 1, 2, ..., n)$, W_n ’e ait teğet vektör alanları ve *n*, W_n ’in normalize edilmiş normal vector alanı olsun. $\left(\overset{*}{\underset{1}{v}}, \overset{*}{\underset{2}{v}}, ..., \overset{*}{\underset{n}{v}}, n\right)$ $(n+1)$ -li şebeke göz önüne alınsın. Genelleştirilmiş kovaryant türev kullanılarak, önce W_n hiperyüzeyinin bir C eğrisinin $\underset{1}{v}$ teğet vector alanına bağlı olarak Frenet formüllerine tekabül eden formüller elde edilmiştir. Sonra, W_n ’de tanımlı $\underset{i}{v}(i = 1, 2, ..., n)$ orthogonal şebekesi yardımıyla iki invariyant tanımlanmıştır.

Anahtar Sözcükler: Weyl uzayı, Eğrilerin şebekesi, Prolonged kovaryand türev.

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1. INTRODUCTION

An n dimensional manifold W_n is said to be Weyl space if it has a symmetric conformal metric tensor g_{ij} and a symmetric connection ∇_k satisfying the compatibility condition given by the equation

$$\nabla_k g_{ij} = 2T_k g_{ij} \tag{1.1}$$

where T_k denotes a covariant vector field. The symmetric tensor g_{ij} is normalization of the form

$$\overset{v}{g}_{ij} = \lambda^2 g_{ij} \tag{1.2}$$

and the covariant vector field T_k is transformed by the rule

$$\overset{v}{T}_k = T_k + \partial_k \ln \lambda \tag{1.3}$$

where λ is a point function on W_n [1].

Let the coordinates of a Weyl hypersurface of n dimensions and its enveloping space of $(n+1)$ dimensions be defined by u^k and x^a, respectively and let

$$\partial_k = \frac{\partial}{\partial u^k}. \tag{1.4}$$

A quantity A is called satellite with weight of k of the fundamental tensor g_{ij}. Under the renormalization of g_{ij} of the form $\overset{v}{g}_{ij} = \lambda^2 g_{ij}$, the quantity A changes according the rule

$$\overset{*}{A} = \lambda^k A. \tag{1.5}$$

The prolonged covariant derivative and the prolonged derivative of the satellite A are defined as follows, respectively

$$\dot{\nabla}_i A = \nabla_i A - k\, T_i\, A \tag{1.6}$$

[2] and

$$\dot{\partial}_i A = \partial_i A - k\, T_i\, A \tag{1.7}$$

[3] where $\nabla_i A$ is usual derivative of the satellite A and $\partial_i A$ is partial derivative of the satellite A.

Let w be any vector belonging to W_n. If w^i are the contravariant components of the vector w, the covariant derivative of w^i with respect to u^k is

$$\nabla_k \ \omega^i = \frac{\partial \omega^i}{\partial u^k} + W^i_{pk} \ \omega^p \tag{1.8}$$

where W^i_{pk} are the coefficients of the connection ∇_k and are defined by the form

$$W^i_{jk} = \left\{ {}^{\,i}_{jk} \right\} - \left(T_k \delta^i_j + T_j \delta^i_k - T_m g_{jk} g^{mi} \right) \tag{1.9}$$

and $\left\{ {}^{\,i}_{jk} \right\}$ are called the second kind Christoffel symbols and are defined as follows

$$\left\{ {}^{\,i}_{jk} \right\} = \frac{1}{2} g^{ir} \left(g_{jr,k} + g_{kr,j} - g_{jk,r} \right). \tag{1.10}$$

Here, the $g_{jr,k}$ is the partial derivative of g_{jr} with respect to u^k.

A Weyl space is shortly denoted by $W_n \left(W^i_{jk}, g_{ij}, T_k \right)$.

Let $W_n \left(g_{ij}, T_k \right)$ be a hypersurface of the Weyl space $W_{n+1} \left(g_{ab}, T_c \right)$ and x^a $\left(a = 1,2,\ldots,n+1 \right)$ and u^i $\left(i = 1,2,\ldots,n \right)$ be the coordinates of $W_{n+1} \left(g_{ab}, T_c \right)$ and $W_n \left(g_{ij}, T_k \right)$, respectively. The metrics of $W_n \left(g_{ij}, T_k \right)$ and $W_{n+1} \left(g_{ab}, T_c \right)$ are connected by the relations

$$g_{ij} = g_{ab} \ x^a_i \ x^b_j \quad \left(j = 1,2,\ldots,n; b = 1,2,\ldots,n+1 \right) \tag{1.11}$$

where x^a_i is the covariant derivative of x^a with respect to u^i.

The prolonged covariant derivative of A with respect to u^k and x^c are $\dot{\nabla}_k A$ and $\dot{\nabla}_c \ A$, respectively. These are related by the conditions

$$\dot{\nabla}_k A = x^c_k \ \dot{\nabla}_c \ A \quad \left(k = 1,2,\ldots,n; c = 1,2,\ldots,n+1 \right). \tag{1.12}$$

Let the normal vector field n^a of $W_n \left(g_{ij}, T_k \right)$ be normalized by the condition $g_{ab} \ n^a \ n^b = 1$. Since the weight of x^a_i is $\{0\}$, the prolonged covariant derivative of x^a_i, relative to u^k, is given by [1]

$$\dot{\nabla}_k x^a_i = \nabla_k x^a_i = \omega_{ik} \ n^a \tag{1.13}$$

where w_{ik} are the coefficients of the second fundamental form of $W_n \left(g_{ij}, T_k \right)$.

On the other hand, it is easy to see that the prolonged covariant derivative of n^a is given by

$$\dot{\nabla}_k n^a = -\omega_{kl} \ g^{il} \ x^a_i \ . \tag{1.14}$$

By means of (1.13), the prolonged covariant derivative of x^j_a is found to be [4]

$$\dot{\nabla}_k x_a^j = \Omega_k^j \, n_a \, . \tag{1.15}$$

Let $\underset{r}{v}^a$ and $\underset{r}{v}^i$ be the contravariant components of the vector field $\underset{r}{v}$ relative to $W_{n+1}(g_{ab}, T_c)$ and $W_n(g_{ij}, T_k)$, respectively. Denoting the components of $\overset{r}{v}$ relative to $W_{n+1}(g_{ab}, T_c)$ and $W_n(g_{ij}, T_k)$ by $\overset{r}{v}_a$ and $\overset{r}{v}_i$ we have [4,5]

$$\underset{r}{v}^a = x_i^a \, \underset{r}{v}^i \ , \ \overset{r}{v}_a = \overset{i}{x}_a \, \overset{r}{v}_i \, . \tag{1.16}$$

The prolonged covariant derivatives of the vector field $\underset{r}{v}$ and its reciprocal $\overset{r}{v}$ are, respectively, given by [5]

$$\dot{\nabla}_k \underset{r}{v}^i = \overset{s}{\underset{r}{T}}{}_k \, \underset{s}{v}^i \ , \ \dot{\nabla}_k \overset{r}{v}_i = -\overset{r}{\underset{s}{T}}{}_k \, \overset{s}{v}_i \, . \tag{1.17}$$

2. THE FORMULAS BELONGING TO THE ORTHOGONAL NET

The prolonged covariant derivative of n^a in the direction of $\underset{1}{v}$ can be written in the form [6,7]

$$\underset{1}{v}^k \, \dot{\nabla}_k \, n^a = k_1 \, \overset{*}{\underset{1}{v}}{}^a \ , \ k_1 = g_{ab} \left(\underset{1}{v}^k \, \dot{\nabla}_k n^a \right) \overset{*}{\underset{1}{v}}{}^b \tag{2.1}$$

in which

$$g_{ab} \, n^a \, \overset{*}{\underset{1}{v}}{}^b = 0, \ g_{ab} \, n^a n^b = 1 \ , \ g_{ab} \, \overset{*}{\underset{1}{v}}{}^a \, \overset{*}{\underset{1}{v}}{}^b = 1 \, . \tag{2.2}$$

We call $\overset{*}{\underset{1}{v}}$ the first tangent vector field and $k_1 \, \overset{*}{\underset{1}{v}}$ the first curvature vector field and k_1 the first curvature of C.

Since

$$g_{ab} \left(\underset{1}{v}^k \, \dot{\nabla}_k n^a \right) n^b = k_1 g_{ab} \, \overset{*}{\underset{1}{v}}{}^a \, n^b = 0, \tag{2.3}$$

n^a is perpendicular to $\underset{1}{v}^k \, \dot{\nabla}_k n^a$.

Taking the prolonged covariant derivative of $\overset{*}{\underset{1}{v}}$ in the direction of $\underset{1}{v}$, we find that

$$\underset{1}{v}^k \, \dot{\nabla}_k \, \overset{*}{\underset{1}{v}}{}^a = \alpha n^a + \beta \, \overset{*}{\underset{2}{v}}{}^a \tag{2.4}$$

where

$$g_{ab}\, n^{a} \overset{*}{\underset{2}{v}}{}^{b} = 0,\; g_{ab} \overset{*}{\underset{1}{v}}{}^{a} \overset{*}{\underset{2}{v}}{}^{b} = 0\,,\; g_{ab} \overset{*}{\underset{2}{v}}{}^{a} \overset{*}{\underset{2}{v}}{}^{b} = 1\,. \tag{2.5}$$

By using (2.2), (2.3) and (2.5) from (2.4), we obtain $\alpha = -k_1$. Putting $\beta = k_2$, (2.4) becomes

$$\underset{1}{v}{}^{k}\, \dot{\nabla}_{k} \overset{*}{\underset{1}{v}}{}^{a} = -k_1 n^{a} + k_2 \overset{*}{\underset{2}{v}}{}^{a}\ . \tag{2.6}$$

We call $\overset{*}{\underset{2}{v}}$ the second tangent vector field and $k_2 \overset{*}{\underset{2}{v}}$ the second curvature vector field and k_2 the second curvature of C.

Since $\overset{*}{\underset{s}{v}}$ is perpendicular to $\overset{*}{\underset{s-1}{v}}$, the equality $g_{ab} \overset{*}{\underset{s}{v}}{}^{a} \overset{*}{\underset{s-1}{v}}{}^{b} = 0$ is satisfied. If we take the prolonged covariant derivative of this equality in the direction of $\underset{1}{v}$, we get

$$\underset{1}{v}{}^{k}\, \dot{\nabla}_{k} \overset{*}{\underset{s}{v}}{}^{a} = -k_{s} \overset{*}{\underset{s-1}{v}}{}^{a} + k_{s+1} \overset{*}{\underset{s+1}{v}}{}^{a} \tag{2.7}$$

We call $\overset{*}{\underset{s+1}{v}}$ the (s+1) th. tangent vector field and $k_{s+1} \overset{*}{\underset{s+1}{v}}$ the (s+1) th. curvature vector field and k_{s+1} the (s+1) th. curvature of C.

Proceeding the same way, we finally obtain the derivative of $\overset{*}{\underset{n}{v}}$ in the form

$$\underset{1}{v}{}^{k}\, \dot{\nabla}_{k} \overset{*}{\underset{n}{v}}{}^{a} = -k_{n} \overset{*}{\underset{n-1}{v}}{}^{a} \tag{2.8}$$

Generally, the expression (2.8) can be written as

$$\underset{1}{v}{}^{k}\, \dot{\nabla}_{k} \overset{*}{\underset{p}{v}}{}^{a} = k_{p+1} \overset{*}{\underset{p+1}{v}}{}^{a} - k_{p} \overset{*}{\underset{p-1}{v}}{}^{a} \tag{2.9}$$

where $k_{n+1} = 0$ and $k_0 = 0$.

We call $\overset{*}{\underset{n}{v}}$ the nth. tangent vector field and $k_n \overset{*}{\underset{n+1}{v}}$ the nth. curvature vector field and k_n the nth. curvature of C.

In this way, we obtain n mutually orthogonal vectors $\overset{*}{\underset{1}{v}}, \overset{*}{\underset{2}{v}}, \ldots, \overset{*}{\underset{n}{v}}$ at a point P of C which satisfy the condition

$$g_{ab} \overset{*}{\underset{r}{v}}{}^{a} \overset{*}{\underset{s}{v}}{}^{b} = \delta_{r}^{s} \quad (r, s = 1, 2, \ldots, n) \tag{2.10}$$

Now, we shall obtain some important results by the first and second order prolonged covariant derivatives of the vector fields of the ennuple:

From (2.7), we have

$$\underset{1}{v}^l \dot{\nabla}_l \overset{*}{\underset{s}{v}}{}^a = -k_s \overset{*}{\underset{s-1}{v}}{}^a + k_{s+1} \overset{*}{\underset{s+1}{v}}{}^a$$

If we take the prolonged covariant derivative of both sides of the above equality in the direction of $\underset{1}{v}$, we have

$$\underset{s}{v}^m \dot{\nabla}_m \left(\underset{1}{v}^l \dot{\nabla}_l \overset{*}{\underset{s}{v}}{}^a \right) = \underset{1}{v}^m \left(\dot{\nabla}_m \underset{1}{v}^l \right) \left(\dot{\nabla}_l \overset{*}{\underset{s}{v}}{}^a \right) + \underset{1}{v}^l \underset{1}{v}^m \dot{\nabla}_m \left(\dot{\nabla}_l \overset{*}{\underset{s}{v}}{}^a \right)$$

$$= k_{s+1} \left(k_{s+2} \overset{*}{\underset{s+2}{v}}{}^a - k_{s+1} \overset{*}{\underset{s}{v}}{}^a \right) + \overset{*}{\underset{s+1}{v}}{}^a \underset{1}{v}^m \left(\dot{\nabla}_m k_{s+1} \right) - k_s \left(k_s \overset{*}{\underset{s}{v}}{}^a - k_{s-1} \overset{*}{\underset{s-2}{v}}{}^a \right)$$

$$- \overset{*}{\underset{s-1}{v}}{}^a \underset{1}{v}^m \left(\dot{\nabla}_m k_s \right) \qquad (2.11)$$

If we multiple (2.11) by $g_{ab} \overset{*}{\underset{s+1}{v}}$, we get

$$g_{ab} \overset{*}{\underset{s+1}{v}}{}^b \underset{1}{v}^m \dot{\nabla}_m \left(\underset{1}{v}^l \dot{\nabla}_l \overset{*}{\underset{1}{v}}{}^a \right) = \underset{1}{v}^m \left(\dot{\nabla}_m k_{s+1} \right) \qquad (2.12)$$

If k_{s+1} is constant, the left hand side of (2.12) is zero and if left hand side of (2.12) is zero,

k_{s+1} is constant.

Hence:

Corollary 2.1

The necessary and sufficient condition that the second order prolonged covariant derivative of the s th. tangent vector field in the direction of $\underset{1}{v}$ be orthogonal to $\overset{*}{\underset{s+1}{v}}$ is that k_{s+1} be constant.

Let us multiple (2.11) by $g_{ab} \overset{*}{\underset{s}{v}}{}^b$. Then we have

$$g_{ab} \left(\underset{1}{v}^m \dot{\nabla}_m \left(\underset{1}{v}^l \dot{\nabla}_l \overset{*}{\underset{s}{v}}{}^a \right) \right) \overset{*}{\underset{s}{v}}{}^b = -\left(k_{s+1}^2 + k_s^2 \right). . \qquad (2.13)$$

If $k_{s+1} = k_s = 0$, the right hand side of (2.13) is zero. That is, the left hand side of (2.13) is zero. If left hand side of (2.13) is zero, $k_s = k_{s+1} = 0$. From this it follows that

Corollary 2.2

The necessary and sufficient condition that the second order prolonged covariant derivative of the s th. tangent vector field be orthogonal to itself is that both $k_s = 0$ and $k_{s+1} = 0$.

If we multiple (2.11) by $g_{ab}\underset{s-1}{\overset{*}{v}}{}^{b}$, we obtain

$$g_{ab}\left(\underset{1}{v}{}^{m}\dot{\nabla}_{m}\left(\underset{1}{v}{}^{l}\dot{\nabla}_{l}\underset{s}{\overset{*}{v}}{}^{a}\right)\right)\underset{s-1}{\overset{*}{v}}{}^{b}=-\underset{1}{v}{}^{m}\dot{\nabla}_{m}k_{s}\,. \tag{2.14}$$

If k_s is constant, the right hand side of (2.14) is zero and conversely, if the left hand side of (2.14) is zero, then k_s is constant.

Hence:

<u>Corollary 2.3</u>

The necessary and sufficient condition that the second order prolonged covariant derivative of the s th. tangent vector field in the direction of $\underset{1}{v}$ be orthogonal to the $(s-1)$th. tangent vector field is that k_s be constant.

If we replace s by $s-1$ in (2.12), we get

$$g_{ab}\underset{s}{\overset{*}{v}}{}^{b}\underset{1}{v}{}^{m}\left(\dot{\nabla}_{m}\left(\underset{1}{v}{}^{l}\dot{\nabla}_{l}\underset{s-1}{\overset{*}{v}}{}^{a}\right)\right)=\underset{1}{v}{}^{m}\dot{\nabla}_{m}k_{s}\,. \tag{2.15}$$

From (2.14) and (2.15), we obtain

$$g_{ab}\underset{s}{\overset{*}{v}}{}^{a}\underset{1}{v}{}^{m}\left(\dot{\nabla}_{m}\left(\underset{1}{v}{}^{l}\dot{\nabla}_{l}\underset{s-1}{\overset{*}{v}}{}^{b}\right)\right)=-g_{ab}\left(\underset{1}{v}{}^{m}\dot{\nabla}_{m}\left(\underset{1}{v}{}^{l}\dot{\nabla}_{l}\underset{s}{\overset{*}{v}}{}^{a}\right)\right)\underset{s-1}{\overset{*}{v}}{}^{b}\,. \tag{2.16}$$

Therefore:

<u>Corollary 2.4</u>

The components of the second order prolonged covariant derivative along the $(s-1)$th. tangent vector field is equal to the negative of the components of the $(s-1)$th. tangent vector field along the s th. tangent vector field.

Since the vector field $\underset{s-1}{\overset{*}{v}}$ is perpendicular to $\underset{s}{\overset{*}{v}}$, $g_{ab}\underset{s-1}{\overset{*}{v}}{}^{a}\underset{s}{\overset{*}{v}}{}^{b}=0$ holds. If we take the prolonged covariant derivative of this equality, we find

$$g_{ab}\underset{s-1}{\overset{*}{v}}{}^{a}\left(\underset{1}{v}{}^{l}\dot{\nabla}_{l}\underset{s}{\overset{*}{v}}{}^{b}\right)+g_{ab}\left(\underset{1}{v}{}^{l}\dot{\nabla}_{l}\underset{s-1}{\overset{*}{v}}{}^{a}\right)\underset{s}{\overset{*}{v}}{}^{b}=0\,. \tag{2.17}$$

Again if we take the prolonged covariant derivative of the same equality, we have

$$g_{ab}\underset{s-1}{\overset{*}{v}}{}^{a}\left(\underset{1}{v}{}^{m}\dot{\nabla}_{m}\left(\underset{1}{v}{}^{l}\dot{\nabla}_{l}\underset{s}{\overset{*}{v}}{}^{b}\right)\right)+2g_{ab}\left(\underset{1}{v}{}^{m}\dot{\nabla}_{m}\underset{s-1}{\overset{*}{v}}{}^{a}\right)\left(\underset{1}{v}{}^{l}\dot{\nabla}_{l}\underset{s}{\overset{*}{v}}{}^{b}\right)+$$

$$g_{ab}\left(\underset{1}{v}{}^{m}\dot{\nabla}_{m}\left(\underset{1}{v}{}^{l}\dot{\nabla}_{l}\underset{s-1}{\overset{*}{v}}{}^{a}\right)\right)\underset{s}{\overset{*}{v}}{}^{b}=0. \tag{2.18}$$

With the help of (2.17), we have

$$g_{ab}\left(\underset{1}{v}^{m}\,\dot{\nabla}_{m}\,\overset{*}{\underset{s-1}{v}}{}^{a}\right)\left(\underset{1}{v}^{l}\,\dot{\nabla}_{l}\,\overset{*}{\underset{s}{v}}{}^{b}\right)=0 \qquad (2.19)$$

From here:
Corollary 2.5

The prolonged covariant derivatives of the consecutive tangent vector fields in the direction of $\underset{1}{v}$ are orthogonal.

Now, if we multiple (2.11) by $g_{ab}\,\overset{*}{\underset{s-2}{v}}{}^{b}$, we obtain

$$g_{ab}\left(\underset{1}{v}^{m}\,\dot{\nabla}_{m}\left(\underset{1}{v}^{l}\,\dot{\nabla}_{l}\,\overset{*}{\underset{s}{v}}{}^{a}\right)\right)\overset{*}{\underset{s-2}{v}}{}^{b}=k_{s}k_{s-1}\,. \qquad (2.20)$$

If k_s or k_{s-1} is zero then the left hand side of (2.20) is zero and if the left hand side of (2.20) is zero, either $k_s=0$ or $k_{s-1}=0$.

Hence:
Corollary 2.6

The necessary and sufficient condition that the second order prolonged covariant derivative of the s th. tangent vector field be orthogonal to $(s-2)$th. tangent vector field is that either $k_s=0$ or $k_{s-1}=0$.

If we multiple (2.11) by $g_{ab}\,\overset{*}{\underset{s+2}{v}}{}^{b}$, we have

$$g_{ab}\left(\underset{1}{v}^{m}\,\dot{\nabla}_{m}\left(\underset{1}{v}^{l}\,\dot{\nabla}_{l}\,\overset{*}{\underset{s}{v}}{}^{a}\right)\right)\overset{*}{\underset{s+2}{v}}{}^{b}=k_{s+1}k_{s+2}\,. \qquad (2.21)$$

If $k_{s+1}=0$ or $k_{s+2}=0$, then the left hand side of (2.21) is zero. Conversely, if the left hand side of (2.21) is zero, then either $k_{s+1}=0$ or $k_{s+2}=0$.

Therefore:

The necessary and sufficient condition that the second order prolonged covariant derivative of the s th. tangent vector field in the direction of $\underset{1}{v}$ be orthogonal to $(s+2)$th. tangent vector field is that either $k_{s+1}=0$ or $k_{s+2}=0$.

If s is replaced by s +2 in (2.20), we have

$$g_{ab}\,\underset{1}{v}^{m}\left(\dot{\nabla}_{m}\left(\underset{1}{v}^{l}\,\dot{\nabla}_{l}\,\overset{*}{\underset{s+2}{v}}{}^{a}\right)\right)\overset{*}{\underset{s}{v}}{}^{b}=k_{s+1}k_{s+2}\,. \qquad (2.22)$$

If we compare (2.21) and (2.22), we get

$$g_{ab}\left(\underset{1}{v}^{m}\,\dot{\nabla}_{m}\left(\underset{1}{v}^{l}\,\dot{\nabla}_{l}\,\overset{*}{\underset{s}{v}}^{a}\right)\right)\overset{*}{\underset{s+2}{v}}^{b}=g_{ab}\left(\underset{1}{v}^{m}\,\dot{\nabla}_{m}\left(\underset{1}{v}^{l}\,\dot{\nabla}_{l}\,\overset{*}{\underset{s+2}{v}}^{a}\right)\right)\overset{*}{\underset{s}{v}}^{b}\,. \tag{2.23}$$

We know that $g_{ab}\,\overset{*}{\underset{s}{v}}^{a}\,\overset{*}{\underset{s+2}{v}}^{b}=0$. If we take the prolonged covariant derivative of both sides equation in the direction of $\underset{1}{v}$, we have $g_{ab}\,\overset{*}{\underset{s}{v}}^{a}\,\underset{1}{v}^{l}\,\dot{\nabla}_{l}\,\overset{*}{\underset{s+2}{v}}^{b}+g_{ab}\left(\underset{1}{v}^{l}\,\dot{\nabla}_{l}\,\overset{*}{\underset{s}{v}}^{a}\right)\overset{*}{\underset{s+2}{v}}^{b}=0$. Again if we take the prolonged covariant derivative of the last equality, we get

$$g_{ab}\,\overset{*}{\underset{s}{v}}^{a}\left(\underset{1}{v}^{m}\,\dot{\nabla}_{m}\left(\underset{1}{v}^{l}\,\dot{\nabla}_{l}\,\overset{*}{\underset{s+2}{v}}^{b}\right)\right)+g_{ab}\left(\underset{1}{v}^{m}\,\dot{\nabla}_{m}\,\overset{*}{\underset{s}{v}}^{a}\right)\left(\underset{1}{v}^{l}\,\dot{\nabla}_{l}\,\overset{*}{\underset{s+2}{v}}^{b}\right)+$$
$$g_{ab}\left(\underset{1}{v}^{l}\,\dot{\nabla}_{l}\,\overset{*}{\underset{s}{v}}^{a}\right)\left(\underset{1}{v}^{m}\,\dot{\nabla}_{m}\,\overset{*}{\underset{s+2}{v}}^{b}\right)\quad+g_{ab}\left(\underset{1}{v}^{m}\,\dot{\nabla}_{m}\left(\underset{1}{v}^{l}\,\dot{\nabla}_{l}\,\overset{*}{\underset{s}{v}}^{a}\right)\right)\overset{*}{\underset{s+2}{v}}^{b}=0\,. \tag{2.24}$$

Using (2.23) and (2.24), we obtain

$$g_{ab}\left(\underset{1}{v}^{m}\,\dot{\nabla}_{m}\,\overset{*}{\underset{s}{v}}^{a}\right)\left(\underset{1}{v}^{l}\,\dot{\nabla}_{l}\,\overset{*}{\underset{s+2}{v}}^{b}\right)=-g_{ab}\,\overset{*}{\underset{s}{v}}^{a}\,\underset{1}{v}^{m}\,\dot{\nabla}_{m}\left(\underset{1}{v}^{l}\,\dot{\nabla}_{l}\,\overset{*}{\underset{s+2}{v}}^{b}\right)=$$
$$-g_{ab}\left(\underset{1}{v}^{m}\,\dot{\nabla}_{m}\left(\underset{1}{v}^{l}\,\dot{\nabla}_{l}\,\overset{*}{\underset{s}{v}}^{a}\right)\right)\overset{*}{\underset{s+2}{v}}^{b} \tag{2.25}$$

From here:
<u>Corollary 2.7</u>

The necessary and sufficient conditions that the prolonged covariant derivative of $\overset{*}{\underset{s}{v}}$ in the direction of $\underset{1}{v}$ be orthogonal to the prolonged covariant derivative of $\overset{*}{\underset{s+2}{v}}$ is that either the second order prolonged covariant derivative of $\overset{*}{\underset{s+2}{v}}$ be orthogonal to $\overset{*}{\underset{s}{v}}$ or the second order prolonged covariant derivative of $\overset{*}{\underset{s}{v}}$ be orthogonal to $\overset{*}{\underset{s+2}{v}}$.

Now, we consider a particular case. For example; the case of $s=1$. Then we obtain from (2.11) that

$$\underset{1}{v}^{m}\,\dot{\nabla}_{m}\left(\underset{1}{v}^{l}\,\dot{\nabla}_{l}\,\overset{*}{\underset{1}{v}}^{a}\right)=k_{2}\left(k_{3}\,\overset{*}{\underset{3}{v}}^{a}-k_{2}\,\overset{*}{\underset{1}{v}}^{a}\right)+\overset{*}{\underset{2}{v}}^{a}\,\underset{1}{v}^{m}\left(\dot{\nabla}_{m}k_{2}\right)-$$
$$k_{1}\left(k_{1}\,\overset{*}{\underset{1}{v}}^{a}\right)-n^{a}\,\underset{1}{v}^{m}\,\dot{\nabla}_{m}k_{1} \tag{2.26}$$

Multiplying (2.26) by $g_{ab} \overset{*}{\underset{3}{v}}{}^{b}$, the following equality is found

$$g_{ab}\left(\underset{1}{v}{}^{m} \dot{\nabla}_{m} \left(\underset{1}{v}{}^{l} \dot{\nabla}_{l} \overset{*}{\underset{1}{v}}{}^{a} \right)\right) \overset{*}{\underset{3}{v}}{}^{b} = k_2 k_3 \,. \tag{2.27}$$

If $k_2 = 0$ or $k_3 = 0$, then the left hand side of (2.40) is zero. Conversely, if the left hand side of (2.27) is zero then either $k_2 = 0$ or $k_3 = 0$.

Therefore:

<u>Corollary 2.8</u>

If $k_2 = 0$ or $k_3 = 0$, the second order prolonged covariant derivative of the first tangent vector field in the direction of $\underset{1}{v}$ is orthogonal to the third tangent vector field. The converse is also true.

If we multiple (2.26) by $g_{ab} \overset{*}{\underset{1}{v}}{}^{b}$, we get

$$g_{ab}\left(\underset{1}{v}{}^{m} \dot{\nabla}_{m} \left(\underset{1}{v}{}^{l} \dot{\nabla}_{l} \overset{*}{\underset{1}{v}}{}^{a} \right)\right) \overset{*}{\underset{1}{v}}{}^{b} = -\left(k_2{}^{2} + k_1{}^{2} \right). \tag{2.28}$$

If $k_2 = 0 = k_1$, then the left hand side of (2.28) is zero. Conversely, if the left hand side of (2.28) is zero, then $k_2 = 0 = k_1$.

Hence:

<u>Corollary 2.9</u>

$k_2 = 0 = k_1$ is the necessary and sufficient condition for the orthogonality of the second order prolonged covariant derivative of the first tangent vector field in the direction of $\underset{1}{v}$ to itself.

If we multiple (2.26) by $g_{ab} \overset{*}{\underset{1}{v}}{}^{b}$, we get

$$g_{ab}\left(\underset{1}{v}{}^{m} \dot{\nabla}_{m} \left(\underset{1}{v}{}^{l} \dot{\nabla}_{l} \overset{*}{\underset{1}{v}}{}^{a} \right)\right) \overset{*}{\underset{2}{v}}{}^{b} = \underset{1}{v}{}^{m} \dot{\nabla}_{m} k_2 \,. \tag{2.29}$$

If $k_2 =$ constant, then the left hand side of (2.29) is zero. Conversely, if the left hand side of (2.29) is zero, we obtain $k_2 =$ constant.

From here:

<u>Corollary 2.10</u>

The necessary and sufficient condition that the second order prolonged covariant derivative of the first tangent vector field in the direction of $\underset{1}{v}$ be orthogonal to the second tangent vector field is that k_2 be constant.

Multiplying (2.26) by $g_{ab}\, n^b$, we have

$$g_{ab}\left(\underset{1}{v}^m \dot{\nabla}_m \left(\underset{1}{v}^l \dot{\nabla}_l \overset{*}{\underset{1}{v}}{}^a\right)\right) n^b = -\underset{1}{v}^m \dot{\nabla}_m k_1 . \tag{2.30}$$

If k_1 is constant, then the left hand side of (2.30) is zero. Conversely, if the left hand side of (2.30) is zero, then k_1 is constant.

Therefore:

Corollary 2.11

$k_1 =$constant is the necessary and sufficient condition that the second order prolonged covariant derivative of the first tangent vector field in the direction of $\underset{1}{v}$ be orthogonal to the normal vector field.

Hence:

If the right hand side of (2.28) is zero, then the right hand side of (2.27), (2.29) and (2.30) are also zero.

From this it follows

Corollary 2.12

If the second order prolonged covariant derivative of the first tangent vector field in the direction of $\underset{1}{v}$ is orthogonal to itself, then it is also orthogonal to the normal vector field, the second tangent vector field and the third tangent vector field.

Multiplying (2.6) by $g_{ab} n^b$, we get

$$g_{ab}\left(\underset{1}{v}^l \dot{\nabla}_l \overset{*}{\underset{1}{v}}{}^a\right) n^b = -k_1 \tag{2.31}$$

If $k_1 = 0$, then the left hand side of (2.31) is zero and the converse of it is also true.

Hence:

Corollary 2.13

The necessary and sufficient condition that the prolonged covariant derivative of the first tangent vector field in the direction of $\underset{1}{v}$ be orthogonal to the normal vector field is that k_1 be zero.

If we multiple (2.6) by $g_{ab}\, \overset{*}{\underset{2}{v}}{}^b$, we obtain

$$g_{ab}\left(\underset{1}{v}^l \dot{\nabla}_l \overset{*}{\underset{1}{v}}{}^a\right) \overset{*}{\underset{2}{v}}{}^b = k_2 . \tag{2.32}$$

If $k_2 = 0$, then the left hand side of (2.32) is zero. The converse is also true. Thus, we have

Corollary 2.14

The necessary and sufficient condition that the first order prolonged covariant derivative of the first tangent vector field in the direction of $\underset{1}{v}$ be orthogonal to the second tangent vector field is that k_2 be zero.

If $k_1 = 0$ and $k_2 = 0$, then the right hand side of (2.31) and (2.32) are zero. In this case, the right hand side of (2.28) is zero.
Therefore, we have

Corollary 2.15

If the second order prolonged covariant derivative of the first tangent vector field is orthogonal to itself, then the first order prolonged covariant derivative of the first tangent vector field is orthogonal to the normal vector field as well as to the second tangent vector field.

We have seen from (2.32) that if $k_2 = 0$ the first order prolonged covariant derivative of the first tangent vector field is orthogonal to the second tangent vector field. The converse of it is also true. But from (2.27) for $k_2 = 0$, it is seen that the second order prolonged covariant derivative of the first tangent vector field is orthogonal to the third tangent vector field. Hence:

Corollary 2.16

If the first order prolonged covariant derivative of the first tangent vector field in the direction of $\underset{1}{v}$ is orthogonal to the second tangent vector field, then the second order prolonged covariant derivative of the first tangent vector field in the direction of $\underset{1}{v}$ is orthogonal to the third tangent vector field.

We know that the equation of (2.28) had been expressed as

$$g_{ab}\left(\underset{1}{v}^m \dot{\nabla}_m\left(\underset{1}{v}^l \dot{\nabla}_l \overset{*}{\underset{1}{v}}{}^a\right)\right)\overset{*}{\underset{1}{v}}{}^b = -\left(k_2{}^2 + k_1{}^2\right). \tag{2.33}$$

If k_2 is constant, then from (2.29) the second order prolonged covariant derivative of the first tangent vector field in the direction of $\underset{1}{v}$ is orthogonal to the second tangent vector field.

If k_1 is constant, then from (2.30) the second order prolonged covariant derivative of the first tangent vector field in the direction of $\underset{1}{v}$ is orthogonal to the normal vector field.

If the conditions $k_1 =$constant and $k_2 =$constant are satisfied, then we obtain from (2.33) that

$$g_{ab}\left(\underset{1}{v}^m \dot{\nabla}_m\left(\underset{1}{v}^l \dot{\nabla}_l \overset{*}{\underset{1}{v}}{}^a\right)\right)\overset{*}{\underset{1}{v}}{}^b = \text{constant.}$$

Corollary 2.17

If the second order prolonged covariant derivative of the first tangent vector field is orthogonal both to the normal vector field and to the second tangent vector, then this and the first tangent vector field cut each other under a constant angle.

3. THE DERIVATIVE FORMULAS FOR A GEODESIC TANGENT

Definition 3.1: Let C be a curve in the Wey hypersurface W_n and let $\underset{1}{v}$ be the tangent vector field of C. If the prolonged covariant derivative of $\underset{1}{v}$ in the direction of itself is zero, then C is called geodesic, i.e.

$$\underset{1}{v}^k \dot{\nabla}_k \underset{1}{v}^l = 0 .$$

Let us consider the geodesic tangent vector field of the curve C at the point P and let us denote it by C_g . Furthermore, let us denote the tangent vector field, the principal normal vector field and the binormal vector field belonging to C_g by $\underset{1}{\overline{v}}, \overline{n}, \overline{n}_2$, respectively

We know from (2.1) that

$$\underset{1}{v}^k \dot{\nabla}_k n^a = k_1 \underset{1}{\overset{*}{v}}{}^a \tag{3.1}$$

We can write

$$\underset{1}{v}^k \dot{\nabla}_k n^a = \underset{1}{\overline{v}}^k \dot{\nabla}_k n^a = \tau_2 \overline{n}_2^a - \tau_1 \underset{1}{\overline{v}}^a \tag{3.2}$$

from Darboux_Ribocour Equations, where τ_1 and τ_2 are the first and the second curvature of the geodesic tangent to the curve C , respectively, τ_1 and τ_2 are the normal curvature and the geodesic torsion of the curve C , respectively.

From (3.1) and (3.2), we get

$$k_1 \underset{1}{\overset{*}{v}}{}^a = \tau_2 \overline{n}_2^a - \tau_1 \underset{1}{\overline{v}}^a . \tag{3.3}$$

If we multiple (3.3) by itself, we obtain

$$k_1^{\,2} = \tau_1^{\,2} + \tau_2^{\,2} . \tag{3.4}$$

Therefore, we can state the following theorem:

Theorem: 3.1.

If any two of the following properties for a curve in a hypersurface W_n of Weyl space W_{n+1} are satisfied, then the third also holds:

$i)$ The first curvature of the geodesic tangent vanishes.

$ii)$ The geodesic torsion of the curve is zero.

$iii)$ The first curvature of the curve C is zero

Theorem: 3.2.

If the curve C is an asymptotic line, then the prolonged covariant derivative of the normal vector field in the direction of $\underset{1}{v}$ is orthogonal to the curve.

Proof: 3.2.
Let C be an asymptotic line. Then the normal curvature of C is zero, that is, $\underset{11}{\kappa} = 0$.
For C_g,

$$\underset{11}{\kappa} = \tau_1 = 0 \tag{3.5}$$

is satisfied.

We know that

$$\underset{11}{\kappa} = -g_{ab}\left(\underset{1}{v}^k \dot{\nabla}_k n^a\right)\overset{*}{\underset{1}{v}}{}^b . \tag{3.6}$$

From (3.5) and (3.6), we see that $\underset{1}{v}^k \dot{\nabla}_k n^a$ is orthogonal to $\overset{*}{\underset{1}{v}}$.
The proof is completed.

On the other hand, we know that $\underset{1}{v}^k \dot{\nabla}_k n^a = \tau_2 \overline{n}_2^a - \tau_1 \underset{1}{\overline{v}}{}^a$ from (3.2). Since $g_{ab}\left(\underset{1}{v}^k \dot{\nabla}_k n^a\right)\underset{1}{\overline{v}}{}^b = 0$, we get

$$\tau_1 = 0 . \tag{3.7}$$

This says

$$k_1 = \tau_2 . \tag{3.8}$$

Also from (3.3)

$$\overset{*}{\underset{1}{v}}{}^a = \overline{n}_2{}^a \tag{3.9}$$

From (3.8):
<u>Corollary 3.1</u>
The product of the prolonged covariant derivative of the normal vector field along asymptotic line by itself is the geodesic torsion of the asymptotic line.

4. ON THE HYPERSURFACES MEETING UNDER A CONSTANT ANGLE

We consider a curve C with the tangent vector field $\underset{1}{v}$ which is common to two hypersurfaces W_n and $\overline{W}_n$. Let n and $\overline{n}$ be the normal vector fields with respect to these hypersurfaces. If these hypersurfaces meet at a constant angle then the following condition is satisfied:

$$\underset{1}{v}^k \dot{\nabla}_k \left(g_{ab} n^a \overline{n}^b\right) = 0 \tag{4.1}$$

From this it follows that

$$g_{ab}\left\{\underset{1}{v}^k \left(\dot{\nabla}_k n^a\right)\right\}\overline{n}^b + g_{ab} n^a \left\{\underset{1}{v}^k \dot{\nabla}_k \overline{n}^b\right\} = 0 . \tag{4.2}$$

Since W_n and $\overline{W}_n$ meet under a constant angle, C is a line of curvature for both hypersurfaces, so that, $\underset{1}{v}^k \dot{\nabla}_k n^a = k_1 \overset{*}{\underset{1}{v}}{}^a$ where $\overset{*}{\underset{1}{v}}{}^a$ are the components of the tangent vector field to the curve with respect to W_{n+1} and k_1^2 is the inner product of the prolonged covariant derivative of the normal vector field n in the direction of $\underset{1}{v}$ by itself.

Also $\underset{1}{\overline{v}}^k \dot{\nabla}_k \overline{n}^a = \overline{k}_1 \overset{*}{\underset{1}{\overline{v}}}{}^a$ where $\overset{*}{\underset{1}{\overline{v}}}{}^a$ are the components of the tangent vector field to the curve with respect to $\overline{W}_{n+1}$ and $\overline{k}_1^2$ is the inner product of the prolonged covariant derivative of the normal vector field $\overline{n}$ in the direction of $\underset{1}{\overline{v}}^k$ by itself.

From the above information and (4.2), we obtain

$$g_{ab}\left(k_1 \overset{*}{\underset{1}{v}}{}^a\right)\cdot \overline{n}^b + g_{ab} n^a \left(\overline{k}_1 \overset{*}{\underset{1}{\overline{v}}}{}^b\right) = 0,$$

(4.3) or

$$\frac{k_1}{\overline{k}_1} = -\frac{g_{ab} n^a \overset{*}{\underset{1}{\overline{v}}}{}^b}{g_{ab} \overset{*}{\underset{1}{v}}{}^a \overline{n}^b} = -\frac{\cos \sphericalangle \left(n^a, \overset{*}{\underset{1}{\overline{v}}}{}^b\right)}{\cos \sphericalangle \left(\overset{*}{\underset{1}{v}}{}^a, \overline{n}^b\right)} \quad . \tag{4.4}$$

We can express this as:

Theorem : 4.1.

If a curve C is common to two hypersurfaces W_n and $\overline{W}_n$ of the Weyl space $W_{n+1}(g_{ab}, T_c)$ such that they meet under a constant angle along C, then $\frac{k_1}{\overline{k}_1}$ is a gauge invariant for $W_{n+1}(g_{ab}, T_c)$.

5. AN INVARIANT ASSOCIATED WITH AN ORTHOGONAL ENNUPLE IN A WEYL HYPERSURFACE

Theorem: 5.1.

The sum $\sum_{r=1}^{n} k_r^2$ where k_r^2 is the inner product of the prolonged covariant derivative of the normal vector field to a Weyl hypersurface in the direction, of the rth vector of an orthogonal ennuple by itself, is an invariant.

Proof:

Let us denote the orthogonal ennuple in W_n by $\left(\underset{1}{v},\underset{2}{v},\ldots\ldots,\underset{n}{v}\right)$. The prolonged covariant derivative of the normal vector field in the directions of the vectors of the orthogonal ennuple can be expressed, by (1.14), as

$$\underset{r}{v}^k\,\dot{\nabla}_k n^a = -\underset{r}{v}^k\,\omega_{kl}\,g^{il}x_i^a \qquad (r=1,2,\ldots,n). \tag{5.1}$$

From this we obtain

$$\begin{aligned}&\sqrt{g_{ab}\left(\underset{r}{v}^k\,\dot{\nabla}_k n^a\right)\left(\underset{r}{v}^l\,\dot{\nabla}_l n^b\right)} = \sqrt{g_{ab}\left(-\underset{r}{v}^k\,\omega_{kj}g^{ij}x_i^a\right)\left(-\underset{r}{v}^l\,\omega_{lm}g^{mt}x_t^b\right)}\\ &= \sqrt{g^{mj}\,\underset{r}{v}^k\,\underset{r}{v}^l\,\omega_{kj}\omega_{lm}}\end{aligned} \tag{5.2}$$

with the help of (1.11) and (5.1). Let us denote this scalar by k_r, i.e. $k_r^2 = g^{mj}\,\underset{r}{v}^k\,\underset{r}{v}^l\,\omega_{kj}\omega_{lm}$.

If we take the sum of the squares with respect to r, we find from (5.2) that

$$\sum_{r=1}^{n} {k_r}^2 = \sum_{r=1}^{n} g^{mj}\,\underset{r}{v}^k\,\underset{r}{v}^l\,\omega_{kj}\omega_{lm} = g^{mj}g^{kl}\omega_{kj}\omega_{lm}\ , \tag{5.3}$$

since $\sum_{r=1}^{n}\underset{r}{v}^k\,\underset{r}{v}^l = g^{kl}$ for the vector fields of an orthogonal ennuple.

This shows that $\sum_{r=1}^{n}{k_r}^2$ is an invariant. The proof of the theorem is completed.

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