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DYNAMIC ANALYSIS OF GEOMETRICALLY NONLINEAR CIRCULAR PLATES ON WINKLER FOUNDATION

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ABSTRACT

Geometrically nonlinear analysis of thin circular plates on Winkler elastic foundations has been studied in this paper. The nonlinear partial differential equations obtained from von Karman's large deflection plate theory have been solved by using the discrete singular convolution (DSC) in the space domain and the harmonic differential quadrature (HDQ) method in the time domain.
Keywords: Discrete singular convolution, plates, nonlinear analysis, winkler elastic foundation.
MSC number/numarası: 65P40, 65Y20, 65D25.

WINKLER ELASTİK ZEMİNE OTURAN DAİRESEL PLAKALARIN GEOMETRİC BAKIMDAN LİNEER OLMAYAN DİNAMİK ANALİZİ

ÖZET

Bu çalışmada Winkler elastik zemine oturan ince dairesel plakların geometrik bakımdan lineer olmayan analizi verilmiştir. Von Karman teorisi ile elde edilen non-lineer denklem, konum değişkeni için ayrık tekil convolution tekniği, zaman değişkeni için harmonik diferansiyel quadrature metodu ile çözülmüştür.
Anahtar Sözcükler: Ayrık tekil convolution, plak, lineer olmayan analiz, Winkler elastik zemin.

1. INTRODUCTION

A number of analytical and numerical studies have been conducted on the linear and nonlinear dynamic analysis of plates [1-8]. Some selected works in this research topic includes those of Sathyamorth [9] and Leissa [10]. In the present work, an approximate numerical solution of the Von Karman-Donnel type governing equations for the geometrically nonlinear analysis of thin circular plates resting on Winkler elastic foundation is presented. For this purpose, DSC and HDQ methods had been used for spatial and temporal discretization of governing differential equations of problem. To the authors' knowledge, it is the first time the discrete singular convolution approach has been successfully applied to a circular plate resting on an elastic foundation problem of the geometrically nonlinear dynamic analysis.

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2. DIFFERENTIAL QUADRATURE (DQ) METHOD

Recently, the methods of DQ and and harmonic differential quadrature (HDQ) have been extended to solve static and dynamic problems in engineering [12-17]. Harmonic differential quadrature has been proposed by Striz et al. [11]. Unlike the differential quadrature that uses the polynomial functions, such as power functions, Lagrange interpolated, and Legendre polynomials as the test functions, harmonic differential quadrature uses harmonic or trigonometric functions as the test functions. Shu and Xue [12] proposed an explicit means of obtaining the weighting coefficients for the HDQ. When the $f(x)$ is approximated by a Fourier series expansion in the form

$$f(x) = c_0 + \sum_{k=1}^{N/2} (c_k \cos\frac{k\pi x}{L} + d_k \sin\frac{k\pi x}{L}) \tag{1}$$

and the Lagrange interpolated trigonometric polynomials are taken as

$$h_k(x) = \frac{\sin\frac{(x-x_0)\pi}{2}\cdots\sin\frac{(x-x_{k-1})\pi}{2}\sin\frac{(x-x_{k+1})\pi}{2}\cdots\sin\frac{(x-x_N)\pi}{2}}{\sin\frac{(x_k-x_0)\pi}{2}\cdots\sin\frac{(x_k-x_{k-1})\pi}{2}\sin\frac{(x_k-x_{k+1})\pi}{2}\cdots\sin\frac{(x_k-x_N)\pi}{2}} \tag{2}$$

for $k = 0,1,2,....,N$. According to the HDQ, the weighting coefficients of the first-order derivatives A_{ij} for $i \neq j$ can be obtained by using the following formula:

$$A_{ij} = \frac{(\pi/2)P(x_i)}{P(x_j)\sin[(x_i - x_j)/2]\pi}; \qquad i, j = 1,2,3,...,N, \tag{3a}$$

where

$$P(x_i) = \prod_{j=1, j\neq i}^{N} \sin\left(\frac{x_i - x_j}{2}\pi\right); \qquad \text{for } j = 1,2,3,...,N. \tag{3b}$$

A natural, an often convenient, choice for sampling points is that of equally spaced point. The equally sampling grid (E-SG) points are given for temporal discretization as;

$$t_j = \frac{j-1}{N-1} \tag{4}$$

3. DISCRETE SINGULAR CONVOLUTION (DSC)

The discrete singular convolutions (DSC) algorithm was originally introduced by Wie [18] as a simple and highly efficient numerical technique. As stated by Wie [19] singular convolutions (SC) are a special class of mathematical transformations, which appear in many science and engineering problems, such as the Hilbert, Abel and Radon transforms. In this paper, details of DC method are not given; interested readers may refer to the works of Wie [20,22] and Wie et. al.[23] who originated the method. The high frequency vibration analysis of plates using DSC algorithm is given by Zhau et al. [24]. Consider a distribution, T and $\eta(t)$ as an element of the space of the test function. A singular convolution can be defined by

$$F(t) = (T * \eta)(t) = \int_{-\infty}^{\infty} T(t-x)\eta(x)dx \tag{5}$$

where $T(t-x)$ is a singular kernel. The DSC algorithm can be realized by using many approximation kernels. The regularized Shannon's kernel (RSK) is given by

$$\delta_{\Delta,\sigma}(x-x_k) = \frac{\sin[(\pi/\Delta)(x-x_k)]}{(\pi/\Delta)(x-x_k)} \exp\left[-\frac{(x-x_k)^2}{2\sigma^2}\right]; \ \sigma>0 \tag{6}$$

where $\Delta=\pi/(N-1)$ is the grid spacing and N is the number of grid points. The parameter σ determines the width of the Gaussian envelop and often varies in association with grid spacing, i.e., $\sigma= r$h. Here r is a parameter chosen in computation. It is also known that the truncation error is very small due to the use of the Gaussian regularizer, the above formulation given by Eq. (6) is practical and has an essentially compact support for numerical interpolation. With a sufficiently smooth approximation, it is more effective to consider a discrete singular convolution

$$F_\alpha(t) = \sum_k T_\alpha(t-x_k) f(x_k) \tag{7}$$

where F_α (t) is an approximation to $F(t)$ and $\{x_k\}$ is an appropriate set of discrete points on which the DSC (5) is well defined. Note that, the original test function $\eta(x)$ has been replaced by f(x). This new discrete expression is suitable for computer realization. The mathematical property or requirement of f(x) is determined by the approximate kernel T_α. In the DSC method, function f (x) and its derivatives with respect to x coordinate at a grid point x_i are approximated by a linear sum of discrete values $f(x_k)$ in a narrow bandwidth $[x-x_M, x+x_M]$. This can be expressed as

$$\left.\frac{d^n f(x)}{dx^n}\right|_{x=x_i} = f^{(n)}(x) \approx \sum_{k=-M}^{M} \delta^{(n)}_{\Delta,\sigma}(x_i - x_k) f(x_k) \tag{8}$$

where superscript n denotes the nth-order derivative with respect to x. The x_k is a set of discrete sampling points centered around the point x, σ is a regularization parameter, Δ is the grid spacing, and 2M+1 is the computational bandwidth, which is usually smaller than the size of the computational domain. The higher order derivative terms $\delta^{(n)}_{\Delta,\sigma}(x-x_k)$ in Eq.(6) are given as below;

$$\delta^{(n)}_{\Delta,\sigma}(x-x_k) = \left(\frac{d}{dx}\right)^n [\delta_{\Delta,\sigma}(x-x_k)] \tag{9}$$

where, the differentiation can be carried out analytically. In Eq.(9) $\delta_\Delta(x-x_k) = \Delta\delta_\alpha(x-x_k)$ and superscript (n) denotes the nth-order derivative, and $2M$+1 is the computational bandwidth which is centered around x and is usually smaller than the whole computational domain. For example the second order derivative at $x=x_i$ of the DSC kernels is given as follows:

$$\delta^{(2)}_{\Delta,\sigma}(x-x_j) = \frac{d^2}{dx^2}\left[\delta_{\Delta,\sigma}(x-x_j)\right]\Big|_{x=x_i} \tag{10}$$

4. GOVERNING EQUATIONS

We consider a thin circular plate resting on Winkler elastic foundation. The geometry of a typical circular plate resting on Winkler elastic foundation is shown in Fig.1. Including the normal inertia and neglecting damping of the foundation, the distributed reaction from the elastic foundation on the plate at any instant of time *t* is given by [7]

$$k_f w + \rho_f h_f w_{tt} \tag{11}$$

where w is the lateral displacement, k_f is the Winkler parameter, ρ_f is the material density of the foundation, h_f is the thickness of the foundation. The general form of the governing equation of a geometrically nonlinear analysis of circular plates on Winkler foundation is expressed by [7]

$$(\nabla^2 - \frac{1}{r^2})\psi + \frac{Eh}{r}(\frac{\partial w}{\partial r})^2 = 0 \tag{12a}$$

$$D\nabla^4 w - \frac{1}{r}\frac{\partial}{\partial r}(\psi \frac{\partial w}{\partial r}) + k_s w + \rho ch \frac{\partial w}{\partial t} + \rho h \frac{\partial^2 w}{\partial t^2} - q = 0 \tag{12b}$$

In order to derive the dimensionless equations governing the axisymmetric large deformations, we introduce the following dimensionless variables and parameters:

$$R = r/a,\ W = w/a,\ \Psi = \frac{1-\upsilon^2}{Eha}\psi,\ P = q\,a^3/D$$

$$K = k_s a^4/D,\ C = c\sqrt{\rho h\, a^4/D},\ \tau = t\sqrt{D/\rho h\, a^4} \tag{13}$$

where D is the flexural rigidity of the plate given by $D = E\,h^3/12(1-\upsilon^2)$, υ Poisson's ratio, q is the uniform static load, c is the damping coefficient, k_s is the stiffness of the foundation, h is the thickness, r is the radius, w is the deflection of the middle surface of the plate, E denote the Young's modulus, ρ is the material density, ψ is the stress function, R is the non-dimensional radius, τ is the non-dimensional time parameter.

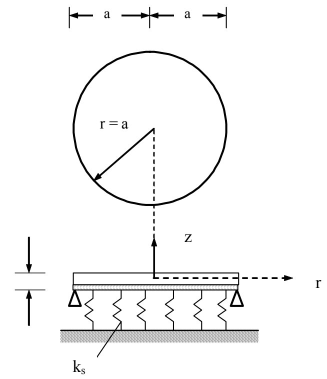

Figure 1. A schematic diagram of a circular plate on an elastic foundation

Then, the equations for the nonlinear dynamic response of thin circular plates are written in the following dimensionless form [7]:

$$R^2(\psi_{,RR}+\frac{1}{R}\psi_{,R}-\frac{1}{R^2}\psi)+\frac{1-\upsilon^2}{2}R(W_{,R})^2=0 \tag{14a}$$

$$R^3(W_{,RRRR}+\frac{2}{R}W_{,RRR}+\frac{1}{R^2}W_{,RR})-12(\frac{a}{h})^2R^2(\psi_{,R}W_{,R}+W_{,RR}\psi)$$
$$+R^3[KW]-R^3[P-CW_{,\tau}-W_{,\tau\tau}]=0 \tag{14b}$$

4.1. Boundary and initial conditions

In the following the DSC and HDQ methods are applied to discretize the derivatives for spatial and time domain in the governing equations, boundary and symmetry conditions and initial conditions. After spatial and time discretization, the governing equations, boundary, symmetry and initial conditions become

$$+R^2\left(\sum_{k=-M}^{M}\delta^{(2)}_{\Delta,\sigma}(k\Delta)\Psi_{i+k,j}+\frac{1}{R}\sum_{k=-M}^{M}\delta^{(1)}_{\Delta,\sigma}(k\Delta)\Psi_{i+k,j}-\frac{1}{R^2}\Psi_{i,j}\right)$$
$$+\frac{1-\upsilon^2}{2}R\left(\sum_{k=-M}^{M}\delta^{(1)}_{\Delta,\sigma}(k\Delta)\Psi_{i+k,j}\right)^2=0 \tag{15a}$$

$$+R^3\left(\sum_{k=-M}^{M}\delta^{(4)}_{\Delta,\sigma}(k\Delta)W_{i+k,j}+\frac{2}{R}\sum_{k=-M}^{M}\delta^{(3)}_{\Delta,\sigma}(k\Delta)W_{i+k,j}+\frac{1}{R^2}\sum_{k=-M}^{M}\delta^{(2)}_{\Delta,\sigma}(k\Delta)W_{i+k,j}\right)$$
$$-12(\frac{a}{h})^2R^2\left(\sum_{k=-M}^{M}\delta^{(1)}_{\Delta,\sigma}(k\Delta)\Psi_{i+k,j}\sum_{k=-M}^{M}\delta^{(1)}_{\Delta,\sigma}(k\Delta)W_{i+k,j}+\sum_{k=-M}^{M}\delta^{(2)}_{\Delta,\sigma}(k\Delta)W_{i+k,j}\Psi_{i,j}\right)$$
$$+R^3\left[KW_{i,j}\right]\quad -R^3\left[P-C\left(T\sum_{j=0}^{N\tau}A_{ij}\,u(\tau_j)\right)-T^2\sum_{j=0}^{N\tau}B_{ij}\,u(\tau_j)\right]=0 \tag{15b}$$

For SS boundary conditions:

$$W_{i+k,N}=0 \tag{16a}$$

$$R^2\left(\sum_{k=-M}^{M}\delta^{(2)}_{\Delta,\sigma}(k\Delta)W_{i+k,N}+\upsilon\sum_{k=-M}^{M}\delta^{(1)}_{\Delta,\sigma}(k\Delta)W_{i+k,N}=0\right) \tag{16b}$$

$$R\sum_{k=-M}^{M}\delta^{(1)}_{\Delta,\sigma}(k\Delta)\Psi_{i+k,N}-\upsilon\Psi_{i+k,N}=0 \tag{16c}$$

For C boundary conditions:

$$W_{i+k,N}=0 \tag{17a}$$

$$\sum_{k=-M}^{M} \delta_{\Delta,\sigma}^{(1)}(k\Delta) W_{i+k,N} = 0 \tag{17b}$$

$$R \sum_{k=-M}^{M} \delta_{\Delta,\sigma}^{(1)}(k\Delta) W_{i+k,N} - \Psi_{i+k,N} = 0 \tag{17c}$$

For symmetry conditions:

$$\sum_{k=-M}^{M} \delta_{\Delta,\sigma}^{(1)}(k\Delta) W_{1+k,j} = 0 \tag{18a}$$

$$\sum_{k=-M}^{M} \delta_{\Delta,\sigma}^{(3)}(k\Delta) W_{1+k,j} = 0 \tag{18b}$$

$$\Psi_{1+k,j} = 0 \tag{18c}$$

For initial conditions:

$$W_{1+k,j} = 0 \tag{19a}$$

$$\sum_{\tau=1}^{N_\tau} A_{jk} W_{1\tau} = 0 \tag{19b}$$

where A_{ij} and B_{ij} are the weighting coefficients for the first- and second-order derivatives for temporal discretization using HDQ, $N\tau$ is the total time steps, and $\delta_{\Delta,\sigma}^{(1)}$, $\delta_{\Delta,\sigma}^{(2)}$, $\delta_{\Delta,\sigma}^{(3)}$ and $\delta_{\Delta,\sigma}^{(4)}$ are coefficients of the regularized Shannon's delta kernel for spatial discretization using DSC.

5. NUMERICAL APPLICATIONS AND RESULTS

In this section, a number of numerical examples are given to illustrate the application of the present method. A uniform step load of infinite duration, sinusoidal loading of finite duration, and triangular load of finite duration have been considered (Figure 2).

Table 1 shows the non-dimensional displacements of clamped circular plates on Winkler foundation for K=100. From the Table 1, it is shown that the convergence of DSC results is very good. By comparing with the results of Nath [7], the DSC results using 16 uniform grid points are very accurate. When the number of grid points is greater than 16, the DSC results are independent of grid. Hence, M=16 value was used during the study .

Figure 3 shows the time-deflection curves of the clamped circular plate for damping coefficient C = 10. The obtained results by HDQ for equally sampling grid (ES-G) and non-equally sampling grid (NES-G) points are shown in this figure. The results given by Nath [7] are also plotted in this figure. The numerical solution of the HDQ method using non-equally sampling grid (NES-G) points is equivalent to the Nath's results.

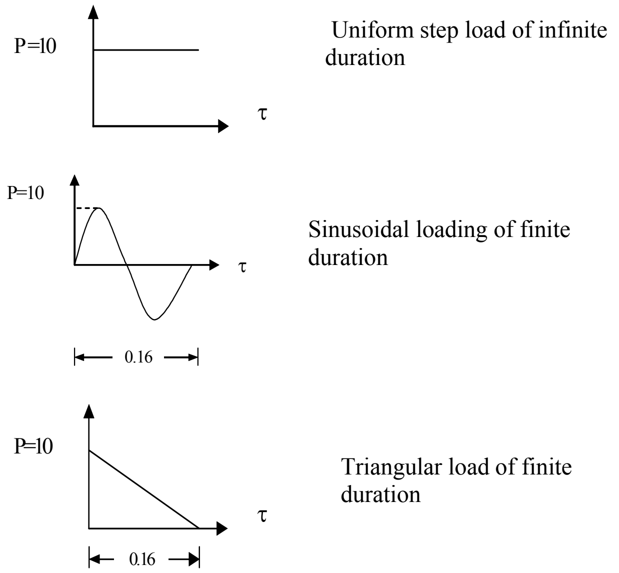

Figure 2. Dynamic loads considered in numerical applications

Table 1. Non dimensional displacement for clamped circular plates

K=100						
Load (P)	DSC results					Ref. 7
	N=11	N=13	N=16	N=18	N=21	
0	0.167	0.09	0.01	0	0	0
4	0.355	0.295	0.245	0.245	0.245	0.243
8	0.543	0.502	0.478	0.480	0.480	0.481
12	0.938	0.817	0.731	0.731	0.735	0.725
16	0.994	0.912	0.890	0.886	0.884	0.894

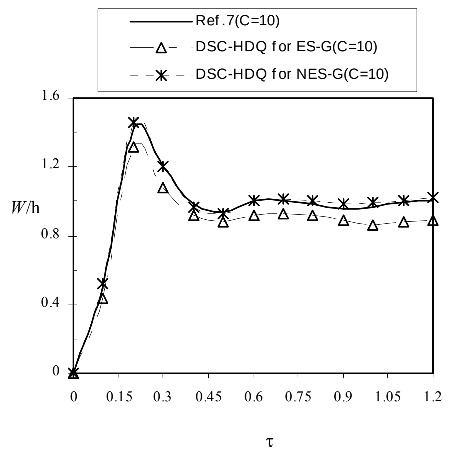

Figure 3. Time- displacement curve for damped dynamic analysis of clamped plates

In figure 4, four-different damping coefficient (*C*=12,16,20,24) is taken into consideration for clamped plate. The damping coefficient *C* has been found to have significant influence on the dynamic response of the circular plates. From these curves given in Figures 3 and 4, it may be concluded that decreasing the damping coefficient, *C* will always result in increased deflection, as expected. Similar results were previously found [5,7].

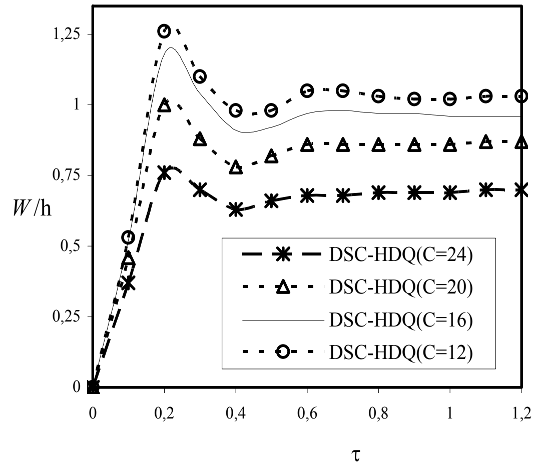

Figure 4. Time- displacement curve of clamped plates for various damped coefficients

The effect of *K* on the response of clamped circular plates resting on Winkler elastic foundation under the step load P=10 is shown in figure 5 together with the results of Nath [7]. The present results are in very good agreement with those of Nath [7] for step load.

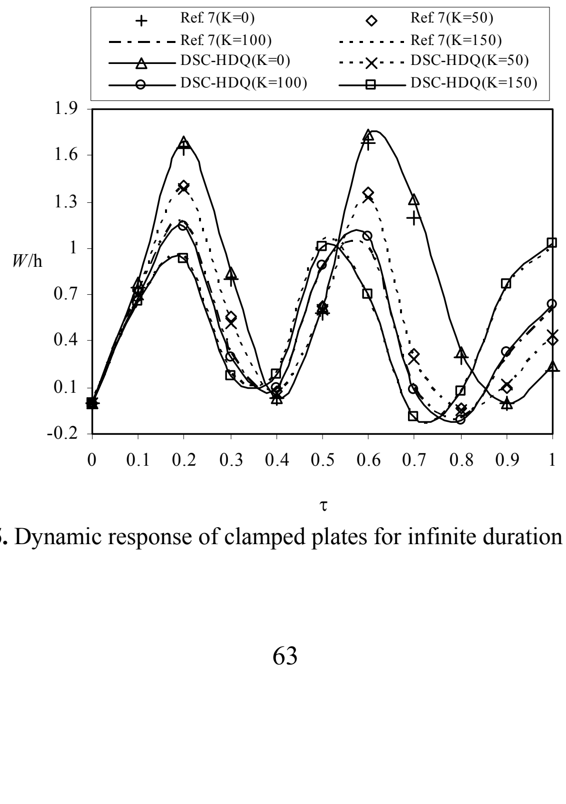

Figure 5. Dynamic response of clamped plates for infinite duration step load

For clamped support condition, the effect of *K* on the response of circular plate under the sinusoidal loading is shown in Figure 6. It is interesting to note, however, that the response to a step load is higher than the response to a sinusoidal load.

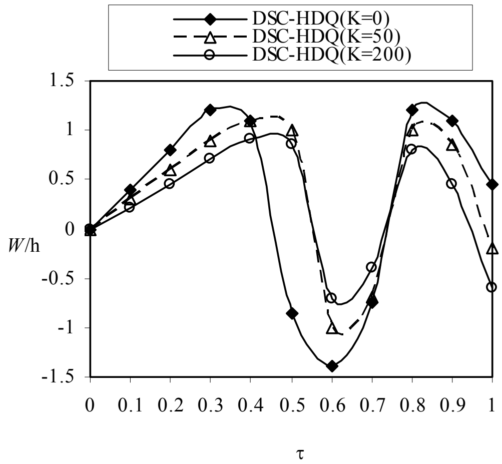

Figure 6. Dynamic response of clamped circular plates for finite duration sinusoidal load

In figure 7, three different types of loading are considered for simply supported square plate. It is shown that the response to a step load is higher than the response to a sinusoidal load and triangular load.

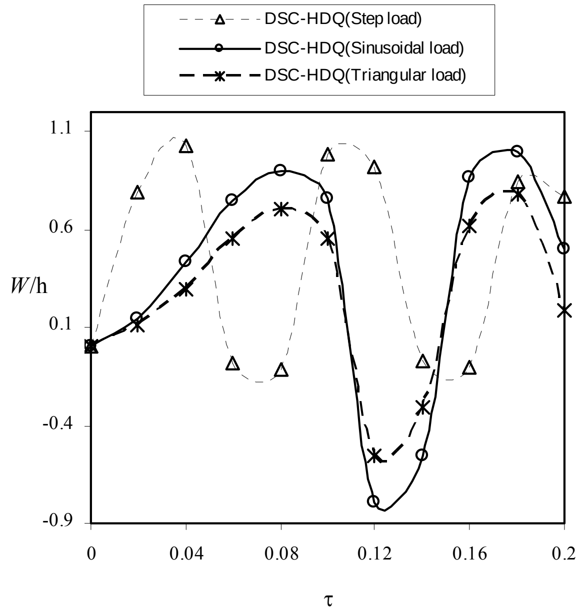

Figure 7. Central deflection versus time for three different types of load (K=150)

6. CONCLUSIONS

The present paper focuses on the application of DSC-HDQ coupled methodolgy. By using the DSC-HDQ coupled methods, geometrically non-linear dynamic analysis is studied for thin circular plates on Winkler foundation. Consequently, by comparing the computed results with those available in published works, the present analysis by the DSC-HDQ methodology is examined and a very good agreement is observed.

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