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ON THE DYNAMICAL PROBLEMS OF THE ELASTIC BODY WITH INITIAL (RESIDUAL) STRESSES

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ABSTRACT

In the present paper the review of the recent investigations regarding the dynamical problems of the bodies with initial stresses are considered. In this case the investigations carried out in the recent six years within the framework of the piecewise homogeneous bodies model with the use of the Three-dimensional Linearized Theory of the Elastic Waves in Initially Stressed Bodies are considered and the main attention is focused on the studies made by the author and his students. The researches on the wave propagation and on the dynamical time-harmonic stress-state problems are reviewed separately. The areas of the further investigations are presented.
Keywords: Initial stress, residual stress, wave dispersion, layered material, fibrous material, time-harmonic stress field, Lamb's problem.
MSC number/numarası: 35L05, 74H10, 74H45, 74J15.

ÖNGERİLMELİ ELASTİK CİSMİN DİNAMİK PROBLEMLERİ HAKKINDA

ÖZET

Bu çalışmada öngerilmeli cisimlerin dinamik problemleriyle ilgili son araştırmaların değerlendirilmesi ele alınmaktadır. Bu durumda öngerilmeli cisimlerde elastik dalgaların üç boyutlu lineerize edilmiş teorisi kullanılarak parçalı homojen cisim modeli çerçevesinde son altı yılda yapılmış araştırmalar ele alınmış ve esas ilgi, yazar ve öğrencileri tarafından yapılmış olan çalışmalara verilmiştir. Dalga yayılımı ve zamana göre harmonik dinamik gerilme durumu problemleri üzerine olan araştırmalar ayrı ayrı ele alınmıştır. İleride yapılması öngörülen araştırma alanları gösterilmektedir.
Anahtar Sözcükler: Öngerilme, dalga dispersiyonu, levhalı malzeme, lifli malzeme, zamana göre harmonik değişen gerilme durumu, Lamb problemi.

1. INTRODUCTION

Elastodynamic problems arise in almost all areas of natural sciences and engineering. As time elapses these problems attract more and more attention of various fundamental and applied areas of science. In this case the intensive development of some fields of the dynamics of the deformed bodies was stimulated by the engineering requirements of the key industries. According to this statement, in the second half of the 20-th century the study of the nonlinear elasdodynamics problems become urgent. In this connection during this time the general nonlinear theory of elastic waves and its various simplified modifications that were oriented toward problems of

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natural science and engineering were intensively developed. In this field a lot of investigations were made and the generalized monographs, such as [1 – 3] have already been published.

An interesting and urgent problem which also applies to the nonlinear dynamical effects in the elastic medium, is the elastodynamics problems for initially stressed bodies. It should be noted that initial stresses occur in the structural elements during their manufacture and assembly, in the Earth's crust under the action of geostatic and geodynamic forces, in composite materials, etc. Therefore results of the investigations of the afore-mentioned elastodynamics problems for initially stressed bodies have a wide range of applications.

At present, by the theory of elastodynamics of the initially stressed bodies is currently meant the linearized theory of the elastodynamics for the initially stressed bodies constructed using the linearization principle from the general nonlinear theory of elasticity or its simplified modifications. In the construction of the field equations of the linearized theory of elastodynamics, one considers two states of a deformable solid. The first is regarded as the initial or unperturbed state and the second is a perturbed state with respect to the unperturbed. By "state of a deformable solid" are meant both motion and equilibrium (as a particular case of motion). It is assumed that all values in a perturbed state can be represented as a sum of the values in the initial state and perturbations. The latter is assumed to be small in comparison with the corresponding values in the initial state. It is also assumed that both initial (unperturbed) and perturbed states are described by the equations of nonlinear deformable solid mechanics. Due to the fact that perturbations are small, the relationships for the perturbed state in the vicinity of appropriate values for the unperturbed state are linearized and then subtracted from them the relationships for the unperturbed state. The result is the linearized equations of the elastodynamics. Since equations contain the initial state variables, the linearized equations describe the influence of the initial stresses on the perturbations.

Thus, within the framework of certain limitations the linearized equations obtained by the above-described manner give the possibility to investigate all kinds of dynamical problems for initially stressed bodies. In this case it is necessary to distinguish, so called, approximate and exact approaches. The approximate approaches are based on the Bernoulli, Kirchhoff-Love and Timoshenko hypotheses and other methods of reducing three-dimensional (two-dimensional) problems to two-dimensional (one-dimensional) ones. It is evident that, the approximate approaches simplify the mathematical solution procedure. However, in many cases the results obtained by employing these approaches may not be acceptable in the qualitative and quantitative sense. For example, the applied theories of rods, plates, and shells describe only few propagating waves (modes). Moreover, within the framework of these approaches it cannot be described the near-surface dynamical processes for the initially stressed bodies.

The investigations carried out by employing TLTEWISB can be divided into two groups. In the first group (the second group) investigations the wave propagation (stress distribution) problems have been studied. Up to now a lot of investigations regarding the first group were made. The review of those was considered in the paper [4-6]. A systematic analysis of the first group of investigations was given in [7-9]. It follows from these references that, before the beginning of the 21st century the investigations regarding: (i) surface wave propagation (except Love waves) in the layered half-space with initial stresses; (ii) the concrete numerical investigations on the wave propagation in the unidirectional fibrous composites with initial stresses are absent, almost completely. Moreover, it follows from the analyses of the foregoing references that until now there are a few studies regarding the above-noted second group investigations. During the recent five years the investigations on the wave propagation problems (i) and (ii) and the investigations on the dynamical stress distribution in the layered materials with initial stresses have been made by the author and his students. The aim of this paper is to consider the review of these investigations and to propose directions of further researches. All results which will be analyzed below are obtained within the framework of the piecewise-homogeneous body model.

Taking the above discussions into account it is preferred to use the exact approach, i.e., so called, Three-dimensional Linearized Theory of Elastic Waves in Initially Stressed Bodies (TLTEWISB) for investigations the dynamical problems of the elastic bodies with initial stresses. It should be noted that for determination of the initial (unperturbed) state in the relatively rigid materials the classical linear theory of elasticity is used. At the same time, the perturbed state is described by the geometrically nonlinear exact equations of the theory of elasticity. By linearizing these equations the afore-mentioned equations of TLTEWISB are obtained. This and other similar versions of the TLTEWISB are analysed in the monographs [7-9].

Below the wave propagation and stress distribution problems will be considered separately.

2. WAVE PROPAGATION (DISPERSION) PROBLEMS

In this section we will consider the wave propagation (dispersion problems regarding the pre-stressed layered half-plane and unidirected fibrous composites. Up to now in this field the results related the half-plane covered with a single layer and the compound circular cylinder were obtained.

2.1. Pre-Stressed Half-Plane Covered With A Single Pre-Stressed Layer

The results for the considered case within the framework of the piecewise homogeneous body model by the use of the Second Version of the Small Deformation Theory (SVSDT) of the TLTEWISB [7-9] were obtained in [10-12]. Here we consider some fragments of these results and for this purpose we start with the mathematical formulation of the problems.

Consider the half-plane covered by the layer with thickness h. With the inter-plane of the layer and half-plane we associate the Lagrangian coordinates $Ox_1x_2x_3$ which in the natural state coincide with the Cartesian coordinates. Note that the covered layer and half-

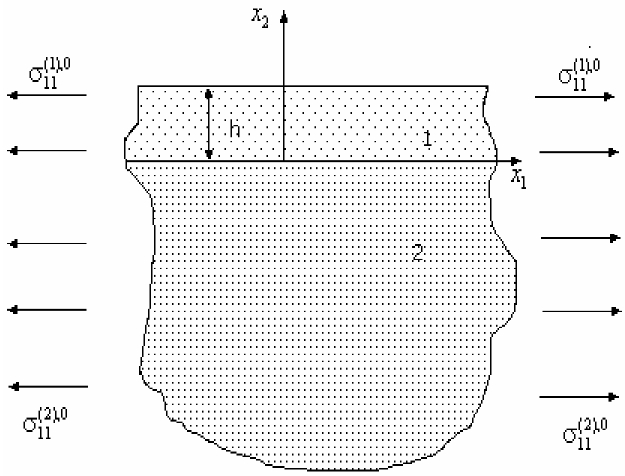


Figure 1. The geometry of the considered stratified half-plane.

plane occupy the regions $\{-\infty < x_1 < +\infty,\ 0 < x_2 < h,\ -\infty < x_3 < +\infty\}$ and $\{-\infty < x_1 < +\infty,\ -\infty < x_2 < 0,\ -\infty < x_3 < +\infty\}$ respectively (Fig.1). Below we will use the following notation: the values related with the layer and half plane are denoted by upper indices (1) and (2) respectively, the values related to the residual (initial) stresses are denoted by upper indices (m),0 where m=1,2.

As it has been noted above, we assume that the initial deformations are small ones and are determined by the use of the classical linear theory of elasticity. Moreover, we assume that these deformations in the covering layer and half space appear separately before contacting these with each other and are determined as follows.

$$\sigma_{11}^{(m),0} = \mathrm{const}_m \neq 0\,,\ \ m = 1,2\,,\ \sigma_{ij}^{(m),0} = 0 \ \text{ for } ij \neq 11\,. \tag{1}$$

Note that all investigations in the [10-12] were made in the plane-strain state in the Ox_1x_2 plane and it was assumed that the stresses arising as a result of the investigated wave propagation in the m-th component of the considered system significantly less than the initial stress $\sigma_{11}^{(m),0}$. Thus, in the [10 -12], within the framework of the foregoing assumptions the dispersion of the waves propagated along the Ox_1 axis is studied. These studies are made by the use of the following equations of motion of TLTEWISB.

$$\frac{\partial\sigma_{11}^{(m)}}{\partial x_1} + \frac{\partial\sigma_{12}^{(m)}}{\partial x_2} + \sigma_{11}^{(m),0}\frac{\partial^2 u_1^{(m)}}{\partial x_1^2} = \rho^{(m)}\frac{\partial^2 u_1^{(m)}}{\partial t^2}\,,$$

$$\frac{\partial\sigma_{12}^{(m)}}{\partial x_1} + \frac{\partial\sigma_{22}^{(m)}}{\partial x_2} + \sigma_{11}^{(m),0}\frac{\partial^2 u_2^{(m)}}{\partial x_1^2} = \rho^{(m)}\frac{\partial^2 u_2^{(m)}}{\partial t^2}\,. \tag{2}$$

In (2) the following notation is used: $\sigma_{ij}^{(m)}$ and $u_i^{(m)}$ are the components of the stress tensor and displacement vector respectively which arise as a result of the wave propagation, $\rho^{(m)}$ is a material density.

It is assumed that on the free face plane of the covering layer the following conditions are satisfied:

$$\left.\sigma_{i2}^{(1)}\right|_{x_2} = 0.\ \ i=1,2. \tag{3}$$

Moreover, it is supposed that the following decay conditions are also satisfied.

$$\sigma_{ij}^{(2)} \xrightarrow[x_2\to-\infty]{} 0\,,\ \ u_i^{(2)} \xrightarrow[x_2\to-\infty]{} 0\ \ . \tag{4}$$

Two types of contact conditions between the covering layer and the half plane are considered, the first of which being the complete contact conditions which can be written as follows:

$$u_i^{(1)}\,|_{x_2=0} = u_i^{(2)}\,|_{x_2=0}\,,\quad \sigma_{i2}^{(1)}\,|_{x_2=0} = \sigma_{i2}^{(2)}\,|_{x_2=0}\,. \tag{5}$$

The second type of contact conditions are the following incomplete contact conditions.

$$u_2^{(1)}\,|_{x_2=0} = u_2^{(2)}\,|_{x_2=0}\ ,\ \sigma_{22}^{(1)}\,|_{x_2=0} = \sigma_{22}^{(2)}\,|_{x_2=0}\,,\ \sigma_{12}^{(1)}\,|_{x_2=0} = 0\,,\ \sigma_{12}^{(2)}\,|_{x_2=0} = 0\ . \tag{6}$$

The mechanical (constitutive) relations of the layer and half-plane materials are given by Murnaghan potential [13]. This potential is determined as follows.

$$\Phi^{(m)} = \frac{1}{2}\lambda^{(m)}\left(A_1^{(m)}\right)^2 + \mu^{(m)}A_2^{(m)} + \frac{a^{(m)}}{3}\left(A_1^{(m)}\right)^3 + b^{(m)}A_1^{(m)}A_2^{(m)} + \frac{c^{(m)}}{3}A_3^{(m)}\,. \tag{7}$$

In (7) $\lambda^{(m)}$ and $\mu^{(m)}$ are Lame's, $a^{(m)}, b^{(m)}$ and $c^{(m)}$ are the 3rd order elasticity constants. Further, $A_1^{(m)}, A_2^{(m)}$ and $A_3^{(m)}$ are the 1st, 2nd and the 3rd algebraic invariants of Green's strain tensor respectively. For the considered case, the expressions of these invariants are

$$A_1^{(m)} = \varepsilon_{11}^{(m)} + \varepsilon_{22}^{(m)}, \quad A_2^{(m)} = \left(\varepsilon_{11}^{(m)}\right)^2 + 2\left(\varepsilon_{12}^{(m)}\right)^2 + \left(\varepsilon_{22}^{(m)}\right)^2,$$

$$A_3^{(m)} = \left(\varepsilon_{11}^{(m)}\right)^3 + 3\left(\varepsilon_{12}^{(m)}\right)^2 \varepsilon_{11}^{(m)} + 3\left(\varepsilon_{12}^{(m)}\right)^2 \varepsilon_{22}^{(m)} + \left(\varepsilon_{22}^{(m)}\right)^3, \tag{8}$$

where

$$\varepsilon_{ij}^{(m)} = \frac{1}{2}\left(\frac{\partial u_i^{(m)}}{\partial x_j} + \frac{\partial u_j^{(m)}}{\partial x_i}\right) \tag{9}$$

From (7)-(9) we obtain the following linearized constitutive relations for the layer and half-plane materials.

$$\sigma_{11}^{(m)} = A_{11}^{(m)}\varepsilon_{11}^{(m)} + A_{22}^{(m)}\varepsilon_{22}^{(m)}, \quad \sigma_{22}^{(m)} = A_{12}^{(m)}\varepsilon_{11}^{(m)} + A_{22}^{(m)}\varepsilon_{22}^{(m)}, \quad \sigma_{12}^{(m)} = 2\mu_{12}^{(m)}\varepsilon_{12}^{(m)}. \tag{10}$$

According to [7 -9], for the considered case, i.e. for the case where the initial stress state is determined within the framework of the geometrical linear theory of elasticity, for the $A_1^{(m)}, A_2^{(m)}$, $A_3^{(m)}$ and $\mu_{12}^{(m)}$ we obtain the following expressions.

$$A_{11}^{(m)} = \lambda^{(m)} + 2\mu^{(m)} + \frac{1}{\mu^{(m)}}\left(2b^{(m)} + c^{(m)}\right)\sigma_{11}^{(m),0} +$$

$$\frac{2\sigma_{11}^{(m),0}}{3K_0^{(m)}}\left[\left(a^{(m)} + b^{(m)}\right) - \left(2b^{(m)} + c^{(m)}\right)\frac{\lambda^{(m)}}{2\mu^{(m)}}\right],$$

$$A_{22}^{(m)} = \lambda^{(m)} + 2\mu^{(m)} + \frac{2\sigma_{11}^{(m),0}}{3K_0^{(m)}}\left[\left(a^{(m)} + b^{(m)}\right) - \left(2b^{(m)} + c^{(m)}\right)\frac{\lambda^{(m)}}{2\mu^{(m)}}\right],$$

$$A_{12}^{(m)} = \lambda^{(m)} + \frac{b^{(m)}}{\mu^{(m)}}\sigma_{11}^{(m),0} + \frac{2\sigma_{11}^{(m),0}}{3K_0^{(m)}}\left[\left(a^{(m)} - b^{(m)}\right)\frac{\lambda^{(m)}}{\mu^{(m)}}\right], \quad K_0^{(m)} = \lambda^{(m)} + \frac{2\mu^{(m)}}{3},$$

$$\mu_{12}^{(m)} = \mu^{(m)} + \frac{b^{(m)}\sigma_{11}^{(m),0}}{3K_0^{(m)}} + \frac{c^{(m)}\sigma_{11}^{(m),0}}{4\mu^{(m)}}\left[\frac{\lambda^{(m)} + \mu^{(m)}}{3K_0^{(m)}}\right]. \tag{11}$$

With the above stated the formulation of the problem is exhausted.

For solution to the formulated problem the displacements are represented as follows

$$u_1^{(m)} = \varphi_1^{(m)}(x_2)\sin(kx_1 - \omega t), \quad u_2^{(m)} = \varphi_2^{(m)}(x_2)\cos(kx_1 - \omega t). \tag{12}$$

The following expressions are obtained for the function $\varphi_2^{(m)}(x_2)$ from the equations (2)-(12).

$$\varphi_2^{(1)}(x_2) = Z_1^{(1)}\exp\left(R_1^{(1)}kx_2\right) + Z_2^{(1)}\exp\left(-R_1^{(1)}kx_2\right) + Z_3^{(1)}\exp\left(R_2^{(1)}kx_2\right) + Z_4^{(1)}\exp\left(-R_2^{(1)}kx_2\right),$$

$$\varphi_2^{(2)}(x_2) = Z_1^{(2)}\exp\left(R_1^{(2)}kx_2\right) + Z_3^{(2)}\exp\left(R_2^{(2)}kx_2\right), \tag{13}$$

where

$$b_{21}^{(m)} = -\frac{A_{11}^{(m)}}{\mu_{11}^{(m)}} - \frac{\sigma_{11}^{(m),0}}{\mu_{12}^{(m)}} + \frac{\rho^{(m)}\omega^2}{\mu_{12}^{(m)}k^2}, \; c_{21}^{(m)} = \frac{-A_{12}^{(m)} - \mu_{12}^{(m)}}{\mu_{12}^{(2)}},$$

$$b_{22}^{(m)} = -\frac{\mu_{12}^{(m)}}{A_{22}^{(m)}} - \frac{\sigma_{11}^{(m),0}}{A_{22}^{(m)}} + \frac{\rho^{(m)}\omega^2}{A_{22}^{(m)}k^2},$$

$$c_{22}^{(m)} = \frac{\mu_{12}^{(m)} + A_{12}^{(m)}}{A_{22}^{(2)}}. \quad B_2^{(m)} = b_{22}^{(m)} + b_{21}^{(m)} - c_{21}^{(m)}c_{22}^{(m)}, \quad C_2^{(m)} = b_{21}^{(m)}b_{22}^{(m)}$$

$$R_1^{(m)} = \sqrt{-\frac{B_2^{(m)}}{2} + \sqrt{\frac{\left(B_2^{(m)}\right)^2}{4} - C_2^{(m)}}}, \; R_2^{(m)} = \sqrt{-\frac{B_2^{(m)}}{2} - \sqrt{\frac{\left(B_2^{(m)}\right)^2}{4} - C_2^{(m)}}} \tag{14}$$

Using the expressions (13) the other sought values are determined from the equations (2), (9) and (10). After doing some mathematical manipulation it is obtained the dispersion relation is obtained from the boundary (3) and contact conditions (5) (for complete contact) or (6) (for incomplete contact). This dispersion relation can be expressed formally as follows

$$\Delta\left(c, kh, \sigma_{11}^{(1),0}, \sigma_{11}^{(2),0}, c_1^{(1)}, c_1^{(2)}, c_2^{(1)}, c_2^{(2)}, a^{(1)}, b^{(1)}, c^{(1)}, a^{(2)}, b^{(2)}, c^{(2)}\right) = 0, \tag{15}$$

where

$$c = \frac{\omega}{k}, \; c_1^{(m)} = \sqrt{\frac{\lambda^{(m)} + 2\mu^{(m)}}{\rho^{(m)}}}, \; c_2^{(m)} = \sqrt{\frac{\mu^{(m)}}{\rho^{(m)}}} \tag{16}$$

Solving equation (15) the dispersion curve

$$c = \psi(kh) \tag{17}$$

is determined. In this case it is assumed that

$$\operatorname{Re} R_1^{(1)} = \operatorname{Re} R_2^{(1)} = 0, \; R_1^{(2)} > 0, \; R_2^{(2)} > 0 \tag{18}$$

It follows from (14) that to satisfy the conditions (18) the following relations must hold

$$B_2^{(1)} > 0, \; C_2^{(1)} > 0, \; C_2^{(2)} < 0, \; B_2^{(2)} < 0 \tag{19}$$

The relations (19) hold when

$$\max\left(\tilde{c}_1^{(1)}, \tilde{c}_2^{(1)}, \tilde{c}_3^{(1)}\right) < c < \min\left(\tilde{c}_1^{(2)}, \tilde{c}_2^{(2)}, \tilde{c}_3^{(2)}\right) \tag{20}$$

where

$$\tilde{c}_1^{(m)} = \sqrt{\frac{A_{11}^{(m)}}{\rho^{(m)}}\left(1 + \frac{\sigma_{11}^{(m),0}}{A_{11}^{(m)}}\right)}, \; \tilde{c}_2^{(m)} = \sqrt{\frac{\mu_{21}^{(m)}}{\rho^{(m)}}\left(1 + \frac{\sigma_{11}^{(m),0}}{\mu_{21}^{(m)}}\right)},$$

$$\tilde{c}_3^{(m)} = \sqrt{\frac{A_{11}^{(m)}}{\rho^{(m)}}\left(\frac{A_{22}^{(m)}}{\mu_{12}^{(m)} + A_{12}^{(m)}} + \frac{\sigma_{11}^{(m),0}}{A_{11}^{(m)}} + \frac{\mu_{12}^{(m)}}{A_{11}^{(m)}}\frac{\mu_{12}^{(m)}}{A_{12}^{(m)} + \mu_{12}^{(m)}} + \frac{A_{12}^{(m)} + \mu_{12}^{(m)}}{A_{11}^{(m)}}\right)}. \tag{21}$$

Consequently, in the case (18) the solution (13) corresponds to such a wave propagation in the layered half-plane that the layer undergoes an oscillatory motion in the Ox_2 direction propagating in the Ox_1 direction with velocity c. According to the above stated, the disturbances in the layer decay exponentially with depth in the half-plane and therefore the wave can be considered as a generalized Rayleigh wave confined to the pre-stressed covering layer.

It should be noted that the considered problem in the case where $\sigma_{11}^{(1),0} = \sigma_{11}^{(2),0} = 0$ was investigated in [14, 15] and also discussed in [16]. According to [14,15], for the case $\sigma_{11}^{(1),0} = \sigma_{11}^{(2),0} = 0$ the function (17) has two branches and it was said that they correspond to the M_1 and M_2 types of propagation. For the M_1 branch the displacement of the layer circumscribes the ellipse similar to the ordinary Rayleigh waves. In this case the M_2 branch leads to an opposite type of motion. Moreover, in [14-16] it had been shown that, the dispersion equation (18) for the case $\sigma_{11}^{(1),0} = \sigma_{11}^{(2),0} = 0$ has infinitely many modes M_{1n} and M_{2n} respectively for the branches M_1 and M_2 unlike ordinary Rayleigh waves, which can propagate only in one mode. There exists a cut-off value of (kh) for each mode, below which unattenuated propagation cannot be excited and the values of this (kh) increase with the mode number.

In [10-12] the corresponding investigations for a stratified half plane were made within the framework of the foregoing statements. In this case the concrete numerical results are obtained for the materials given in Table 1.

Table 1. The values of elastic constants (after [5]) of the selected materials

Materials	ρ gr/cm^3	$\lambda \times 10^{-4}$ MPa	$\mu \times 10^{-4}$ MPa	$a \times 10^{-5}$ MPa	$b \times 10^{-5}$ MPa	$c \times 10^{-5}$ MPa
Steel 3	7.795	9.26	7.75	-2.35	-2.75	-4.90
Bronze	7.20	8.16	3.84	1.20	-3.10	4.80
Brass 59-1	7.20	9.49	4.47	-0.70	2.70	-3.40
Brass 62	7.20	9.49	4.47	-2.80	-2.10	-3.20
Acrylic Plastic	1.16	0.404	0.19	2.68×10^{-3}	-3.12×10^{-2}	-6.77×10^{-2}

Note that in the paper [11] the above-formulated problem was studied for the case where the influence of the third order elastic constants is not taken into account, i.e. it was assumed that $a^{(m)} = b^{(m)} = c^{(m)} = 0.0$ in (7) and (11). The influence of these constants on the considered wave dispersion was analyzed in the paper [10]. Moreover note that in [10, 11] the complete contact conditions (5) were considered. The influence of the incompleteness of the contact conditions (i.e. the conditions (6)) on the wave propagation were studied in [12]. Here we consider some fragments of these results and for this purpose, as in [10], we introduce the parameters $\psi^{(1)} = \sigma_{11}^{(1),0} / \mu^{(1)}$, $\psi^{(2)} = \sigma_{11}^{(2),0} / \mu^{(2)}$. Consider two cases: I. $\psi^{(1)} > 0$, $\psi^{(2)} = 0$ and II. $\psi^{(1)} = 0$, $\psi^{(2)} < 0$. Moreover, introduce the notation $\eta = (\bar{c} - c)/c$, where c is the wave propagation velocity under $\psi^{(1)} = \psi^{(2)} = 0$ and $\bar{c}$ is the wave propagation velocity under $\psi^{(1)} \neq 0$ or $\psi^{(2)} \neq 0$. Consider the pair of materials *bronze(covering layer) + steel (half-plan)*. Fig. 2 shows the dispersion curves for the first and second mode of the M_1 and M_2 branches in the case where $\psi^{(1)} = \psi^{(2)} = 0$. The influence of the pre-stretching of the covering layer and the third order elastic constants of the layer material on the wave propagation velocity is illustrated by the graphs given in Fig. 3. At the same time, Fig. 4 shows the influence of the initial compression of the half plane the dispersion of the generalized Rayleigh wave. In these figures the graphs regarding the afore-mentioned first and second branches of the first (second) mode are represented by letters a and b (c and d).

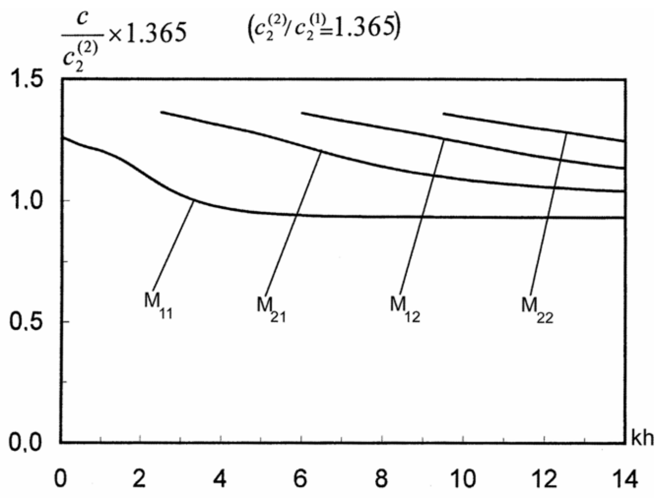


Figure 2. Dispersion curves for the pair of materials bronze + steel.

Note that in [10-12] besides the pair of materials considered here the pairs *brass 59-1 + steel*, *brass 62 + steel*, *acrylic plastic + steel* are also considered. The systematic analysis of these results was made in [17].

According to the numerical results attained in the investigations [10-12, 17] the following conclusions were drawn.

For all the considered materials which are selected for the stratified layer if the influence of the third order elastic constants occurring as a result of the initial stretching (tension) of this layer is ignored the velocity of the generalized Rayleigh wave propagation increases.

It was established that as a result of the incompleteness of the contact conditions the wave propagation velocity decreases.

Under taking the third order elastic constants into account the character of the influence of the above-mentioned initial stretching (tension) of the stratified layer on the velocity of the generalized Rayleigh wave propagation is different for various materials. For example, for the pairs of materials such as *bronze + steel* and *brass 59-1 + steel* this influence has only a quantitative character, i.e. the accounting of the third order elastic constants causes an increase for the pair of the materials *bronze + steel* , but a decrease for the pair of the materials *brass 59-1 + steel* in the generalized Rayleigh wave propagation velocity with respect to that obtained under the case where these constants are ignored. However, for the pairs of materials *brass 62 + steel* and *acrylic plastic + steel* the mentioned influence has not only a quantitative, but also a qualitative character, i.e. as a result of the accounting of the third order elastic constants the velocity of the considered wave decreases under initial stretching of the stratified layer. Moreover, for the pair of the materials *acrylic plastic + steel* the character of this influence depends also on the values of kh.

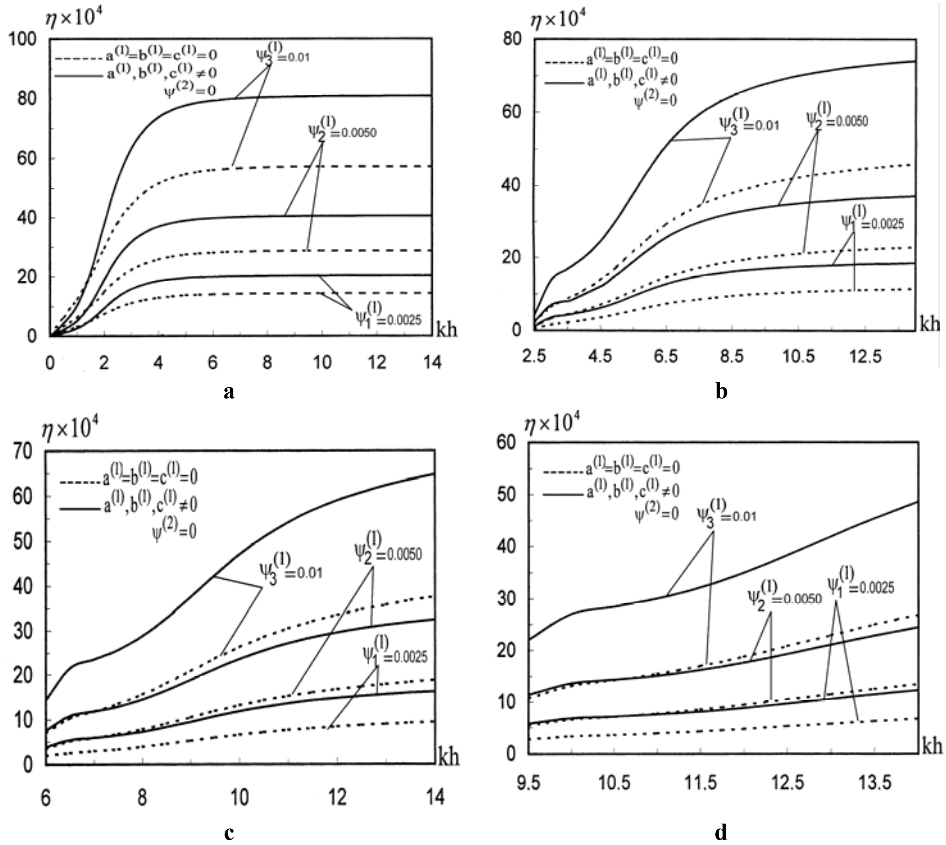


Figure 3. The influence of the pre-stretching of the stratified layer on the dispersion of the generalized Rayleigh wave for the pair materials bronze + steel

With the above-stated we exhaust the consideration of the results attained for the surface wave propagation in the stratified half-plane with initial stresses. Furthermore, this kind of investigations can be developed for the pre-stressed half plane covered with the many-layered pre-stretched plate. Moreover, these investigations can be developed for the finite initial strain state.

2.2. Axisymmetric Longitudinal Wave Propagation In Pre-Stresses Compound Circular Cylinders

This problem was investigated in the paper [18]. The investigations were made within the framework of the piecewise homogeneous body model with the use of the first variant of the theory of small initial strains of the TLTEWISB. The problem considered in [18] is of significance both from the viewpoint of the wave propagation in the many-layered cylindrical bodies and from the viewpoint of the wave propagation in the unidirected fibrous composites under low concentration of the fibers. It is evident that the latter case can be realized as the thickness of the covering hollow cylinder is more significant than the radius of the inner whole cylinder. We here consider the formulation of the problem and some fragments of the results obtained in [18].

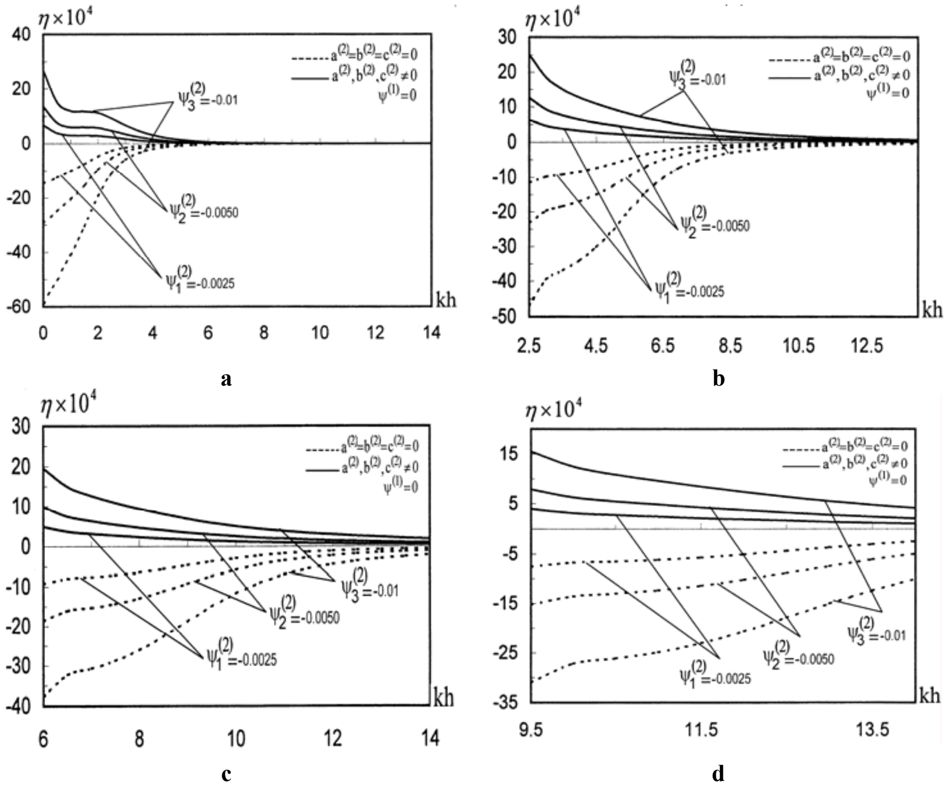


Figure 4. The influence of the initial compression of the half plane on the dispersion of the wave for considered pair of materials

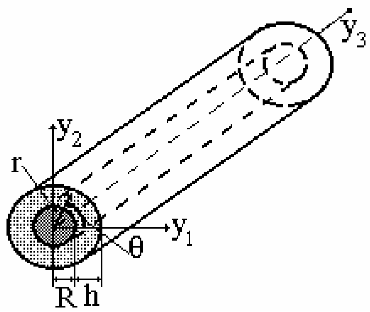


Figure 5. The geometry of the considered compound cylinder.

Assume that in the natural state the inner solid cylinder radius is R (Fig.5) and the thickness of the external hollow cylinder is h. In the natural state we determine the position of the points of the cylinders by the Lagrangian coordinates in the Cartesian system of coordinates $Oy_1y_2y_3$ as well as in the cylindrical system of coordinates $Or\theta y_3$. Assume that the cylinders have infinite length in the direction of the Oy_3 axis. We aim that the cylinders (before the compounding) be stretched separately along the Oy_3 axis and in each of them the homogeneous axisymmetric initial stress state be appear. However, the results which will be discussed below can also be related to the case where the cylinders are stretched after the compounding. In this case as a result of the difference of Poisson's coefficients of the cylinders' materials the inhomogeneous initial stresses acting on the areas which are parallel to the Oy_3 axis arise. But the values of these inhomogeneous stresses are much smaller than the values of the homogeneous initial stresses acting on the areas which are perpendicular to the Oy_3 axis. Therefore, according to [9], the inhomogeneous initial stresses can be neglected under consideration.

With the initial state of the cylinders we associate the Lagrangian cylindrical system of coordinates $O'r'\theta'y'_3$ and Cartesian system of coordinates $O'y'_1y'_2y'_3$. The values related to the inner solid cylinder and external hollow cylinder are denoted by upper indices (1) and (2) respectively. Furthermore, the values related to the initial state are denoted by the upper index "0". Thus, according to the above-stated the initial state in the cylinders can be written as follows:

$$u_m^{(k),0} = (\lambda_m^{(k)} - 1)y_m\,,\quad \lambda_m^{(k)} = \text{const}\,,\ \lambda_1^{(k)} = \lambda_2^{(k)}\,,\quad m = 1,2,3\,;\quad k = 1,2\,, \tag{22}$$

where $\lambda_m^{(k)}$ is the elongation along the Oy_m axis.

In [18] within the above-stated the wave propagation in the $O'y'_3$ direction in the compound cylinder is investigated. As it has been noted above, this investigation was made by the use of coordinates r' and y'_3 in the framework, the first variant of the theory of small initial strains of the TLTEWISB. According to this variant of the small deformation theory, it is assumed that the values $\delta_m^{(k)} = 1 - \lambda_m^{(k)}$ and shears are smaller than unity and thus can be neglected under linearization procedure. How this theory is constructed was analysed in [7-9] in detail.

It follows from (22) that

$$y'_i = \lambda_{\underline{i}}^{(k)} y_{\underline{i}}\,,\ r' = \lambda_1^{(k)} r\,,\quad R' = \lambda_1^{(1)} R\,,\ h' = \lambda_1^{(2)} h\,. \tag{23}$$

Below the values related to the system of coordinates associated with initial state, i.e. with $O'y'_1y'_2y'_3$ or with $O'r'\theta'y'_3$ are denoted by the upper prime.

Thus, according to [7-9], we write the basic relations of the TLTEWISB for axisymmetrical case.

The equation of motion:

$$\frac{\partial}{\partial r'}T'^{(\underline{k})}_{r'r'} + \frac{\partial}{\partial y'_3}T'^{(\underline{k})}_{r'3} + \frac{1}{r'}(T'^{(\underline{k})}_{r'r'} - T'^{(\underline{k})}_{\theta'\theta'}) = \rho'^{(\underline{k})}\frac{\partial^2}{\partial t^2}u'^{(\underline{k})}_{r'}\ ,$$

$$\frac{\partial}{\partial r'}T'^{(\underline{k})}_{3r'} + \frac{1}{r'}T'^{(\underline{k})}_{3r'} + \frac{\partial}{\partial y'_3}T'^{(\underline{k})}_{33} = \rho'^{(\underline{k})}\frac{\partial^2}{\partial t^2}u'^{(\underline{k})}_{3}\,, \tag{24}$$

The mechanical relations:

$$T'^{(\underline{k})}_{r'r'} = \omega'^{(\underline{k})}_{1111}\frac{\partial u'^{(\underline{k})}_{r'}}{\partial r'} + \omega'^{(\underline{k})}_{1122}\frac{u'^{(\underline{k})}_{r'}}{r'} + \omega'^{(\underline{k})}_{1133}\frac{\partial u'^{(\underline{k})}_{3}}{\partial y'_3}\,,$$

$$T'^{(k)}_{\theta'\theta'} = \omega'^{(k)}_{2211} \frac{\partial u'^{(k)}_{r'}}{\partial r'} + \omega'^{(k)}_{2222} \frac{u'^{(k)}_{r'}}{r'} + \omega'^{(k)}_{2233} \frac{\partial u'^{(k)}_{3}}{\partial y'_3} ,$$

$$T'^{(k)}_{r'3} = \omega'^{(k)}_{1313} \frac{\partial u'^{(k)}_{r'}}{\partial y'_3} + \omega'^{(k)}_{1331} \frac{\partial u'^{(k)}_{3}}{\partial r'} , \ T'^{(k)}_{3r'} = \omega'^{(k)}_{3113} \frac{\partial u'^{(k)}_{r'}}{\partial y'_3} + \omega'^{(k)}_{3131} \frac{\partial u'^{(k)}_{3}}{\partial r'} ,$$

$$T'^{(k)}_{33} = \omega'^{(k)}_{3311} \frac{\partial u'^{(k)}_{r'}}{\partial r'} + \omega'^{(k)}_{3322} \frac{u'^{(k)}_{r'}}{r'} + \omega'^{(k)}_{3333} \frac{\partial u'^{(k)}_{3}}{\partial y'_3} . \tag{25}$$

In (24) and (25) through $T'^{(k)}_{r'r'}, \ldots, T'^{(k)}_{33}$ the perturbation of the components of Kirchoff stress tensor are denoted, notation $u'^{(k)}_{r'}$, $u'^{(k)}_{3}$ shows the perturbation of the components of the displacement vector. The constants $\omega'^{(k)}_{1111}, \ldots, \omega'^{(k)}_{3333}$, $\rho^{(k)}$ in (24), (25) are determined through the mechanical constants of the cylinders' materials and through the initial stress state. Note that for the considered initial stress state the expression of these constants is given through the expression of those in the system of coordinates $Or\theta y_3$ (we denote them by $\omega^{(k)}_{1111}, \ldots, \omega^{(k)}_{3333}$) with the following formulae:

$$\omega'^{(k)}_{1111} = \lambda^{(k)2}_{1} \omega^{(k)}_{1111}, \ \omega'^{(k)}_{1331} = \lambda^{(k)2}_{1} \omega^{(k)}_{1331}, \ \omega'^{(k)}_{3333} = \lambda^{(k)2}_{3} \omega^{(k)}_{3333}, \ \omega'^{(k)}_{3113} = \lambda^{(k)2}_{3} \omega^{(k)}_{3113},$$

$$\rho'^{(k)} = \frac{\rho^{(k)}}{\lambda^{(k)}_{1}\lambda^{(k)}_{2}\lambda^{(k)}_{3}}, \ \omega'^{(k)}_{3113} = \lambda^{(k)2}_{3} \omega^{(k)}_{3113}, \ \omega'^{(k)}_{1133} = \lambda^{(k)}_{1}\lambda^{(k)}_{3} \omega^{(k)}_{1133}, \ \omega'^{(k)}_{1313} = \lambda^{(k)}_{1}\lambda^{(k)}_{3} \omega^{(k)}_{1313} \tag{26}$$

It is assumed that the elasticity relations of the cylinder's materials are given by Murnagan potential (7). For the problem considered in [18] the expressions of the algebraic invariants are the following:

$$A_1 = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{33}, A_2 = \varepsilon^2_{rr} + 2\varepsilon^2_{r3} + \varepsilon^2_{\theta\theta} + \varepsilon^2_{33}, \ A_3 = \varepsilon^3_{rr} + \varepsilon^3_{\theta\theta} + \varepsilon^3_{33} + 3\varepsilon^2_{r3}(\varepsilon_{rr} + \varepsilon_{33}) . \tag{27}$$

In (27) ε_{rr}, $\varepsilon_{\theta\theta}$, ε_{33} and ε_{r3} are the components of the Green's strain tensor and these components are determined through the components of the displacement vector by the following formulae:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} + \frac{1}{2}\left(\frac{\partial u_r}{\partial r}\right)^2 + \frac{1}{2}\left(\frac{\partial u_3}{\partial r}\right)^2, \varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{2}\frac{u^2_r}{r^2},$$

$$\varepsilon_{r3} = \frac{1}{2}\left(\frac{\partial u_3}{\partial r} + \frac{\partial u_r}{\partial y_3} + \frac{\partial u_r}{\partial r}\frac{\partial u_r}{\partial y_3} + \frac{\partial u_3}{\partial r}\frac{\partial u_3}{\partial y_3}\right), \quad \varepsilon_{33} = \frac{\partial u_3}{\partial y_3} + \frac{1}{2}\left(\frac{\partial u_3}{\partial y_3}\right)^2 + \frac{1}{2}\left(\frac{\partial u_r}{\partial r}\right)^2 . \tag{28}$$

In this case the components σ_{ij} of the symmetric stress tensor are determined as follows:

$$\sigma_{rr} = \frac{\partial}{\partial \varepsilon_{rr}}\Phi, \ \sigma_{\theta\theta} = \frac{\partial}{\partial \varepsilon_{\theta\theta}}\Phi, \ \sigma_{33} = \frac{\partial}{\partial \varepsilon_{33}}\Phi, \ \sigma_{r3} = \frac{\partial}{\partial \varepsilon_{r3}}\Phi . \tag{29}$$

Note that the expressions (25)-(29) are written in the arbitrary system of cylindrical coordinate system without any restriction related to the association of this system to the natural or initial state of the considered compound cylinders.

For the considered case the relations between the perturbation of the Kirchhoff stress tensor and the perturbation of the components of the ordinary symmetric tensor of stress can be written as follows:

$$T'^{(k)}_{r'r'} = \lambda_1^{(k)} \sigma'^{(k)}_{r'r'} , \quad T'^{(k)}_{\theta'\theta'} = \lambda_1^{(k)} \sigma'^{(k)}_{\theta'\theta'} , \quad T'^{(k)}_{r'3} = \sigma'^{(k)}_{r'3} + \sigma^{(k),0}_{33} \frac{\partial u'^{(k)}_{r'}}{\partial y'_3} ,$$

$$T'^{(k)}_{3r'} = \sigma'^{(k)}_{3r'} , \; T'^{(k)}_{33} = \lambda_3^{(k)} \sigma'^{(k)}_{33} + \sigma^{(k),0}_{33} \frac{\partial u'^{(k)}_{3}}{\partial y'_3} . \tag{30}$$

According to [7-9], by linearization of the equations (29) and taking (25), (30) and (22) into account the following expressions are obtained for the constants $\omega^{(k)}_{1111}, \ldots, \omega^{(k)}_{3333}$ in (25):

$$\omega^{(k)}_{1111} = \omega^{(k)}_{2222} = \lambda_1^{(k)2} A^{(k)}_{11} , \; \omega^{(k)}_{3333} = \lambda_3^{(k)2} A^{(k)}_{33} + \sigma^{(k),0}_{33} , \; \omega^{(k)}_{1133} = \omega^{(k)}_{2233} = \lambda_1^{(k)} \lambda_3^{(k)} A^{(k)}_{13} ,$$

$$\omega^{(k)}_{1122} = \lambda_1^{(k)2} A^{(k)}_{12} , \; \omega^{(k)}_{3113} = \omega^{(k)}_{3223} = \lambda_1^{(k)2} \mu^{(k)}_{13} + \sigma^{(k),0}_{33} , \; \omega^{(k)}_{1331} = \omega^{(k)}_{2332} = \lambda_3^{(k)2} \mu^{(k)}_{13} ,$$

$$\omega^{(k)}_{1221} = \omega^{(k)}_{2112} = \lambda_1^{(k)2} \mu^{(k)}_{12} , \; \omega^{(k)}_{1313} = \omega^{(k)}_{2323} = \lambda_1^{(k)} \lambda_3^{(k)} \mu^{(k)}_{13} , \; \omega^{(k)}_{1213} = \lambda_1^{(k)2} \mu^{(k)}_{12} , \tag{31}$$

where

$$\lambda_1^{(k)2} = \lambda_2^{(k)2} = 1 - \frac{\lambda^{(k)}}{3K_0^{(k)} \mu^{(k)}} \sigma^{(k),0}_{33} , \quad \lambda_3^{(k)2} = 1 + \frac{2\left(\lambda^{(k)} + \mu^{(k)}\right)}{3K_0^{(k)} \mu^{(k)}} \sigma^{(k),0}_{33} ,$$

$$A^{(k)}_{11} = (\lambda^{(k)} + \mu^{(k)}) \left[1 + \frac{\sigma^{(k),0}_{33}}{\left(\lambda^{(k)} + 2\mu^{(k)}\right) 3K_0^{(k)}} \left(2a^{(k)} - \frac{\lambda^{(k)} - \mu^{(k)}}{\mu^{(k)}} 2b^{(k)} - \frac{\lambda^{(k)}}{\mu^{(k)}} c^{(k)} \right) \right] ,$$

$$A^{(k)}_{33} = (\lambda^{(k)} + \mu^{(k)}) \left[1 + \frac{\sigma^{(k),0}_{33}}{\left(\lambda^{(k)} + 2\mu^{(k)}\right) 3K_0^{(k)}} \left(2a^{(k)} + \frac{2\lambda^{(k)} + 3\mu^{(k)}}{\mu^{(k)}} 2b^{(k)} + \frac{\lambda^{(k)} + \mu^{(k)}}{\mu^{(k)}} 2c^{(k)} \right) \right] ,$$

$$\mu^{(k)}_{13} = \mu^{(k)} \left[1 + \frac{\sigma^{(k),0}_{33}}{3K_0^{(k)} \mu^{(k)}} \left(b^{(k)} + \frac{1}{4} \frac{\lambda^{(k)} + 2\mu^{(k)}}{\mu^{(k)}} c^{(k)} \right) \right] ,$$

$$A^{(k)}_{13} = \lambda^{(k)} \left[1 + \frac{\sigma^{(k),0}_{33}}{3K_0^{(k)} \lambda^{(k)}} \left(2a^{(k)} + \frac{\lambda^{(k)} + 2\mu^{(k)}}{\mu^{(k)}} b^{(k)} \right) \right] ,$$

$$A^{(k)}_{12} = \lambda^{(k)} \left[1 + \frac{\sigma^{(k),0}_{33}}{3K_0^{(k)} \lambda^{(k)}} \left(a^{(k)} - \frac{\lambda^{(k)}}{\mu^{(k)}} b^{(k)} \right) \right] , \; \mu^{(k)}_{12} = \mu^{(k)} \left[1 + \frac{\sigma^{(k),0}_{33}}{3K_0^{(k)} \lambda^{(k)}} \left(b^{(k)} - \frac{\lambda^{(k)}}{2\mu^{(k)}} c^{(k)} \right) \right] \tag{32}$$

Thus, the propagation of wave in the considered compound cylinder was investigated by the use of the equations (24)-(25), (31) and (32). In this case it is assumed that the following contact and boundary conditions are satisfied.

$$T'^{(1)}_{r'r'}\Big|_{r'=R'} = T'^{(2)}_{r'r'}\Big|_{r'=R'} , \; T'^{(1)}_{r'3}\Big|_{r'=R'} = T'^{(2)}_{r'3}\Big|_{r'=R'} , \; u'^{(1)}_{r'}\Big|_{r'=R'} = u'^{(2)}_{r'}\Big|_{r'=R'} ,$$

$$u'^{(1)}_{3}\Big|_{r'=R'} = u'^{(2)}_{3}\Big|_{r'=R'} , \quad T'^{(2)}_{r'r'}\Big|_{r'=R'+h'} = 0 , \quad T'^{(2)}_{r'3}\Big|_{r'=R'+h'} = 0 . \tag{33}$$

With the above-stated we exhaust the formulation of the problem and the consideration of the governing field equations.

The solution to the formulated problem is presented as

$$u'^{(m)}_3 = \varphi^{(m)}_3(r')\sin(ky'_3-\omega t)\,,\; u'^{(m)}_{r'} = \varphi^{(m)}_{r'}(r')\cos(ky'_3-\omega t)\,. \tag{34}$$

After some mathematical manipulations, the functions $\varphi^{(m)}_3(r')$ and $\varphi^{(m)}_{r'}(r')$ in (34) are determined from the foregoing corresponding equations. The expressions of these functions contain unknown constants. From the conditions (33) the homogeneous algebraic system of equations are obtained for these constants and by the usual procedure, the dispersion equation is attained from this system equation.

Table 2. The values of elastic constants of selected materials

Materials	Density	Young's moduli	Pois.'s ratio	Third order elastic constants		
Tungsten (W)	$\rho_W \times 10^{-3} =$ $19.3\,\mathrm{kg/m^3}$	$E_W \times 10^{-4} =$ 34.3 MPa	$\nu_W =$ 0.28	$a^{(1)}_W \times 10^{-5} =$ -10.75 MPa	$b^{(1)}_W \times 10^{-5} =$ -14.3 MPa	$c^{(1)}_W \times 10^{-5} =$ -49.6 MPa
Alumin. (Al)	$\rho_{Al} \times 10^{-3} =$ $2.77\,\mathrm{kg/m^3}$	$E_{Al} \times 10^{-4} =$ 7.28 MPa	$\nu_{Al} =$ 0.30	$a^{(2)}_{Al} \times 10^{-5} =$ 0.62 MPa	$b^{(2)}_{Al} \times 10^{-5} =$ -0.49 MPa	$c^{(2)}_{Al} \times 10^{-5} =$ -3.43 MPa

Here we consider some fragments of the numerical results attained for the case where the material of the external hollow cylinder is aluminium (Al), but the material of the internal cylinder is tungsten (W). The values of the elastic constants of the selected materials are given in Table 2. Introduce the parameters

$$\psi^{(1)} = \frac{\sigma^{(1),0}_{33}}{\mu^{(1)}}\,,\; \psi^{(2)} = \frac{\sigma^{(2),0}_{33}}{\mu^{(2)}} \tag{35}$$

for estimation of the initial stresses.

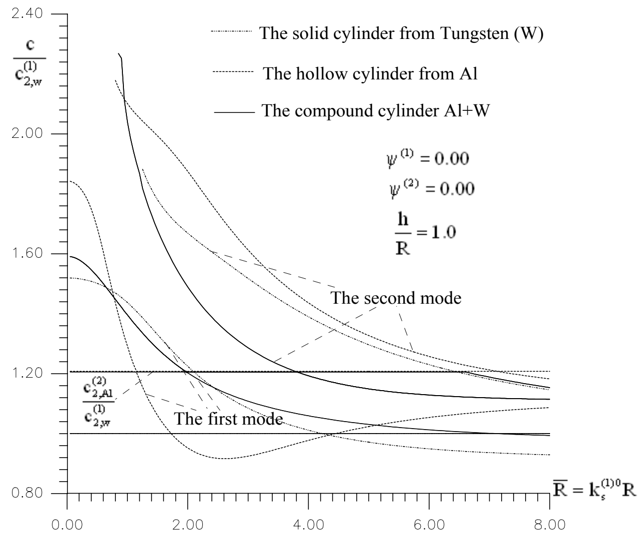


Figure 6. The dispersion of the first and second modes of the compound cylinders under h/R

In Figs. 6, 7 the dispersion curves for the first and second modes of the compound cylinder are given under $h/R = 1.0$ and 5.0 respectively. At the same time, in these figures the corresponding dispersion curves for the cylinder of tungsten and for the hollow cylinder of aluminium are also given for comparison.

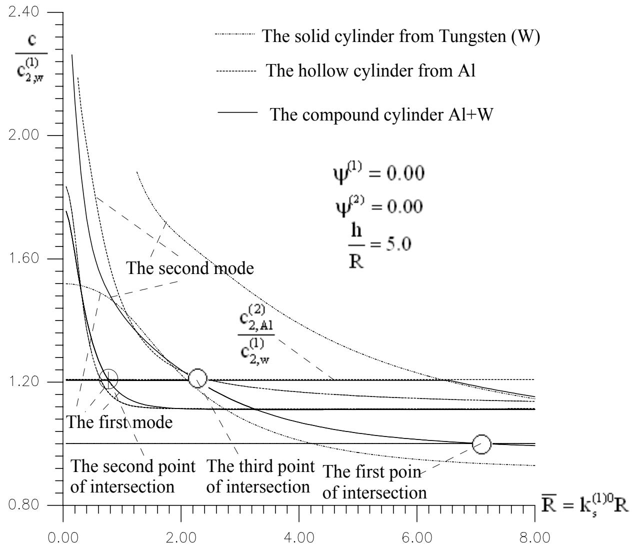


Figure 7. The dispersion curves of the first and second modes of the compound cylinders under $h/R = 5.0$.

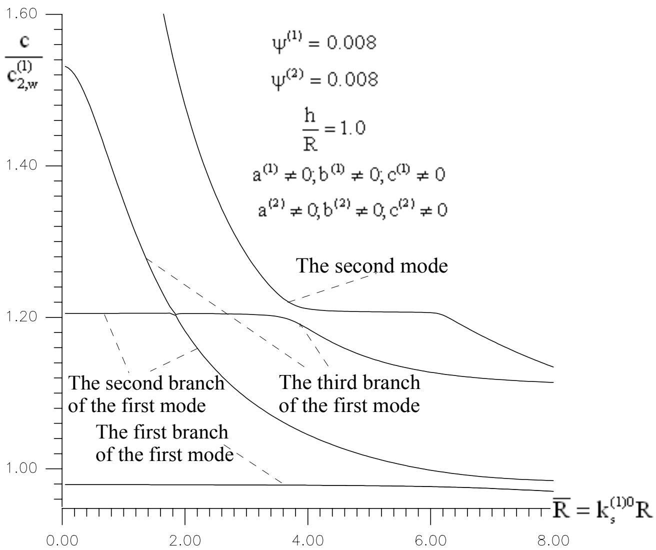


Figure 8. The influence of the initial stretching of the internal solid and external hollow cylinders on the dispersion curves of the compound cylinders

For the determination of the character of the influence of the initial stresses on the dispersion curves the following three cases are considered: $\{\psi^{(1)} = 0.008,\ \psi^{(2)} = 0.00\}$ (call it case I), $\{\psi^{(1)} = 0.00,\ \psi^{(2)} = 0.008\}$ (call it case II) and $\{\psi^{(1)} = 0.008,\ \psi^{(2)} = 0.008\}$ (call it case III). The numerical results obtained in the paper [18] show that in the case I the dispersion curves are divided into the branches around only the first intersection point (Fig.7); in case II the separation of the dispersion curves from the branches takes place around only the second and third intersection point (Fig. 7). However, in the case III the dispersion curves are divided into the branches around all the fixed intersection points (Fig. 7).

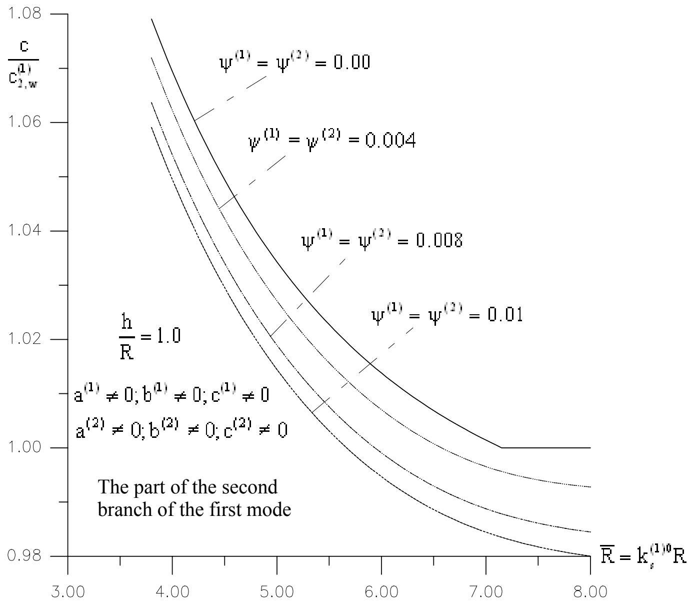


Figure 9. The influence of the initial stretching on the part of second branch of the first mode of the dispersion curves

It follows from the numerical results presented in [18] that under the absence of the initial stresses, $c = c_2$ is a root of the dispersion equation for the solid cylinder from homogeneous materials. If the considered cylinder consists of a certain number of components (materials), then $c = c_2^{(n)}$ (where n is the number of components) are roots of the dispersion equation under the absence of the initial stresses. But, under the existence of the initial stretching in the cylinder the dispersion equation does not have such roots and as a result of both this situation and the separation around the “intersection points” the new branches of the dispersion curves arise. Therefore, the influence of the initial stresses on the wave propagation velocity (in the quantitative sense) is estimated according to these branches.

As an example, in Fig. 8 the afore-mentioned branches are shown. Moreover, Fig. 9 shows the effect of the initial stretching of the components of the compound cylinder on the part of the second branches. Moreover, Fig. 9 shows the influence of the initial stretching of the components of the compound cylinder on the part of the second branch of the first mode of the dispersion curves. At the same time, in the paper [18] it is established that in the case $h/R = 8.0$ can be related to the infinite body containing a single cylinder, or to the unidirected fibrous composite with low concentration of the fibres. Further, the investigations started in the paper [18] can be developed for the unidirected fibrous composites with high concentration of the fibres.

With the above-stated we exhaust the consideration of the recent investigation on the wave propagation problems in the pre-stressed bodies.

3. TIME-HARMONIC STRESS-DISTRIBUTION (FREQUENCY-RESPONSE) PROBLEMS

One of the first attempts in this field was made in the papers [19-21]. Note that in [19,20] the nonstationary Lamb's problem [22] for incompressible half-plane with finite initial strain was considered. But in the paper [21] the investigation [19,20] was developed for compressible half-plane. These investigations were made by employing the Laplace integral transformation with respect to time and some expressions were obtained for the stresses and displacements through which the influence of the initial stresses on the dynamical behaviour of the considered half plane is analyzed. After these studies, during twenty years (from 1980 till the beginning of the 21st century) the investigations in the considered field were absent almost completely. In the beginning of the present century the study of the considered type problems was continued with the papers [23,24] in which the time harmonic Lamb's problem was studied for the stratified half-plane with the initial stresses. The investigations were made in the framework of the piecewise homogeneous bodies model by the use of the SVSIDT of the TLTEWISB. According to SVSIDT the initial strain-stress state in the layer and half-plane was determined by expressions (1) and the equations (2) satisfied within each component of the considered system. It is assumed that on the free face plane of the covering layer the following boundary conditions are satisfied.

$$\sigma_{22}^{(1)}\Big|_{x_2=h} = P_0\delta(x_1)e^{i\omega t}, \quad \sigma_{x_2=h}^{(1)}\Big| = 0 \ . \tag{36}$$

Moreover, it is assumed that between the covering layer and half-plane there exist complete contact conditions (5). In this case the decay conditions (4) are replaced by the following ones

$$\left\{\left|\sigma_{ij}^{(2)}\right|;\left|u_i^{(2)}\right|\right\} < M = const. \tag{37}$$

In other words, it is assumed that as $x_2 \to -\infty$ there is no reflection, which means all waves travel in the negative x_2 direction (Fig. 1). At the same time, the linearly elastic material of the covering layer and the half-plane is supposed to be homogeneous and isotropic and the elasticity relations of these materials are written as follows.

$$\sigma_{ij}^{(m)} = \lambda^{(m)}\theta^{(m)}\delta_i^j + 2\mu^{(m)}\varepsilon_{ij}^{(m)}, \ \theta^{(m)} = \varepsilon_{11}^{(m)} + \varepsilon_{22}^{(m)} \ . \tag{38}$$

Thus, within the framework of the afore-mentioned assumptions in the papers [23,24] the solution to the boundary value problem determined by the equations (2), (36), (5), (37), (38) and (9) the representation

$$u_1^{(m)} = \frac{\partial\varphi^{(m)}}{\partial x_1} + \frac{\partial\psi^{(m)}}{\partial x_2}, \ u_2^{(m)} = \frac{\partial\varphi^{(m)}}{\partial x_2} - \frac{\partial\psi^{(m)}}{\partial x_1} \ . \tag{39}$$

By direct verification it is proven that the potentials $\varphi^{(m)}$ and $\psi^{(m)}$ must satisfy the equations

$$\nabla^2\varphi^{(m)} + \frac{\sigma_{11}^{(m)0}}{\lambda^{(m)} + 2\mu^{(m)}}\frac{\partial^2\varphi^{(m)}}{\partial x_1^2} = \frac{1}{(c_1^{(m)})^2}\frac{\partial^2\varphi^{(m)}}{\partial t^2},$$

$$\nabla^2\psi^{(m)} + \frac{\sigma_{11}^{(m)0}}{\mu^{(m)}}\frac{\partial^2\psi^{(m)}}{\partial x_1^2} = \frac{1}{(c_2^{(m)})^2}\frac{\partial^2\psi^{(m)}}{\partial t^2}. \tag{40}$$

At $\sigma_{11}^{(m),0} = 0$, the equations (40) coincide with those in the classical linear theory of elastodynamics [16].

According to the problem formulation the sought values are presented as $g(x_1, x_2, t) = \bar{g}(x_1, x_2)e^{i\omega t}$ and the exponential Fourier integral transformation is applied with respect to x_1. The inverse transformation of the amplitude $g(x_1, x_2)$ (the over bar is omitted) is determined numerically for which the corresponding algorithm is developed.

Thus, within the framework of the afore-mentioned assumptions and by employing the above-described solution method in the paper [22] the distribution of the normal stress $\sigma_{22}(x_1,0)h/P_0 = \sigma_{22}^{(1)}(x_1,0)h/P_0 = \sigma_{22}^{(2)}(x_1,0)h/P_0$ with respect to x_1/h was studied for various values of the frequency ω and initial stress $\sigma_{11}^{(1),0}/\mu^{(1)}$. Note that such an investigation for the distribution of the shear stress $\sigma_{12}(x_1,0)h/P_0$ was made in the paper [24]. Moreover, note that in the papers [23,24] the numerical investigations were made for the case where $\sigma_{11}^{(2),0}/\mu^{(2)} = 0$ and it is assumed that the stiffness of the covering layer is greater than that of the half-plane. Therefore, in the paper [25] the investigation was developed for the case where the elasticity modulus of the pre-stretching covering layer is smaller than that of the half-plane. It is evident that the case considered in the paper [25] is more complicated than that considered in the papers [23, 24] because the case considered in [25] is suitable for considering the propagation of the above-discussed generalized Rayleigh wave. In [25] the numerical results are obtained for the case where $\psi^{(1)} = \sigma_{11}^{(1)0}/\mu^{(1)} > 0$, $\psi^{(2)} = \sigma_{11}^{(2)0}/\mu^{(2)} = 0$, $\nu^{(1)} = \nu^{(2)} = 0.25$ ($\nu^{(1)}(\nu^{(2)})$ is the Poisson ratio of the layer (half-plane) material, $\rho_0^{(1)} = \rho_0^{(2)}$, $E^{(1)} < E^{(2)}$. The dimensionless frequency $\Omega = \omega h/c_2^{(1)}$ was introduced. In [25] the algorithm for obtaining the numerical results is also developed and the numerical results regarding the distribution of the stresses $\sigma_{22}(x_1,0)h/P_0$, $\sigma_{12}(x_1,0)h/P_0$ with respect to x_1/h for various values of Ω and $E^{(1)}/E^{(2)}$ are presented.

In the paper [26] the investigations were developed for the case where the free face plane of the covering layer is subjected to a uniformly distributed harmonic load acting on a strip extending to infinity in the direction of Ox_3 axis (Fig. 1) and of width $2a$ in the direction of the Ox_1 axis. The plane strain state in the Ox_1x_2 plane is analyzed. In this case the boundary conditions (36) are replaced by the following ones

$$\sigma_{22}^{(1)}\Big|_{x_2=h} = \begin{cases} \dfrac{P_0}{2a}e^{i\omega t} & \text{for } |x_1| \le a \\ 0 & \text{for } |x_1| > a \end{cases}, \quad \sigma_{x_2=h}^{(1)}\Big| = 0 . \tag{41}$$

For the solution to the considered problem the method proposed in [23-25] is employed. In [26] the numerical results are presented within the framework of the assumptions and notation used in [25] and the main attention in these results is focused on the influence of the new problem parameter a/h on the dependencies between $\sigma_{22}(x_1,0)h/P_0$ and x_1/h for the various values of Ω and $E^{(1)}/E^{(2)}$. Moreover, the parameter $\psi_{22} = ((\sigma_{22}(x_1,0)|_{\eta_1=0} - \sigma_{22}(x_1,0)|_{\eta_1>0}) \times 10^3)/P_0$ (where $\eta_1 = \sigma_{11}^{(1)0}/E^{(1)}$) is introduced and the distribution of the ψ_{22} with respect to x_1/h is also studied. The numerical results are presented under $\sigma_{11}^{(2)0}/E^{(2)} = 0$ for both $E^{(1)}/E^{(2)} \geq 1$ and $E^{(1)}/E^{(2)} < 1$ cases. Note that the results obtained for the case $E^{(1)}/E^{(2)} = 1$, $\eta_1 = \sigma_{11}^{(1)0}/E^{(1)} = 0$ approach the regarding ones of the corresponding static problem [27] as $\Omega \to 0$. As an example in Figs. 10 and 11 some fragments of the afore-mentioned results are presented.

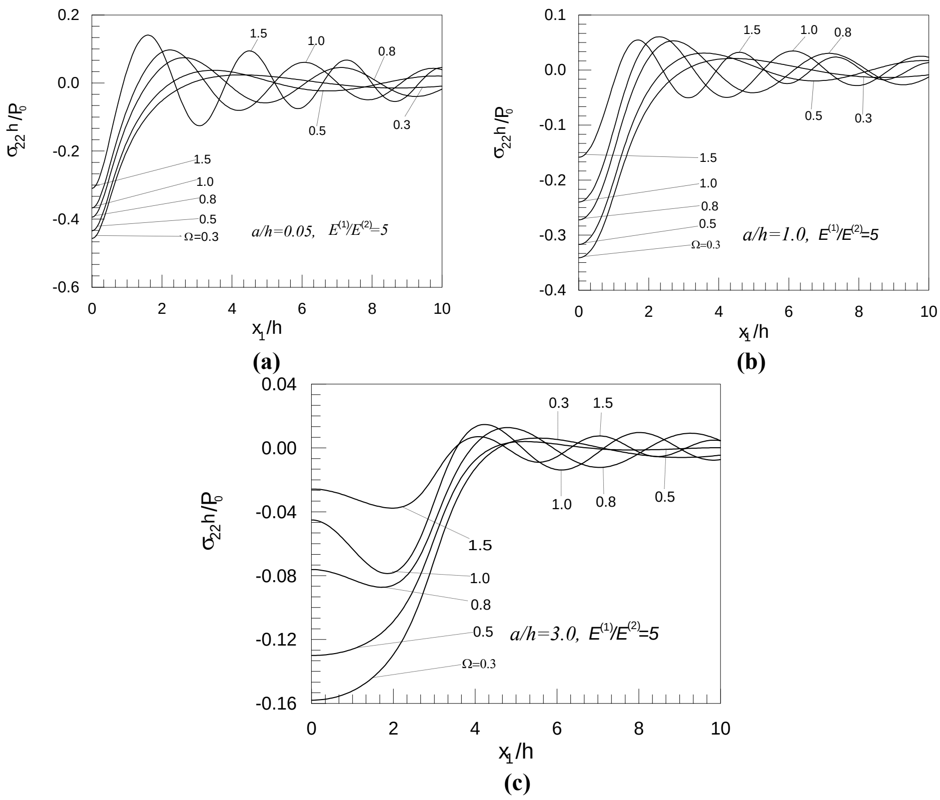


Figure 10. The distribution of $\sigma_{22}h/P_0$ with respect to x_1/h

Note that the study [23 – 26] is performed for the case where the time-harmonic lineal-located or uniformly distributed load is perpendicular to the face plane of the covering layer. Moreover, in [23 – 26] the numerical results are given for some selected discrete values of the dimensionless frequency Ω. In the paper [28] the investigations started in [23 – 26] were developed for the case where on the free face plane of the covering layer the arbitrary inclined lineal located time-harmonic forces act. In other words, in the paper [28] the boundary conditions (36) are replaced by the following ones.

$$\sigma_{12}^{(1)}\Big|_{x_2=h} = -P_0\delta(x_1)e^{i\omega t}\cos\alpha\ ,\ \ \sigma_{22}^{(1)}\Big|_{x_2=h} = -P_0\delta(x_1)e^{i\omega t}\sin\alpha\ , \tag{42}$$

where α is an angle between the Ox_1 axis and the external force vector direction, $\delta(x_1)$ is the Dirac delta function.

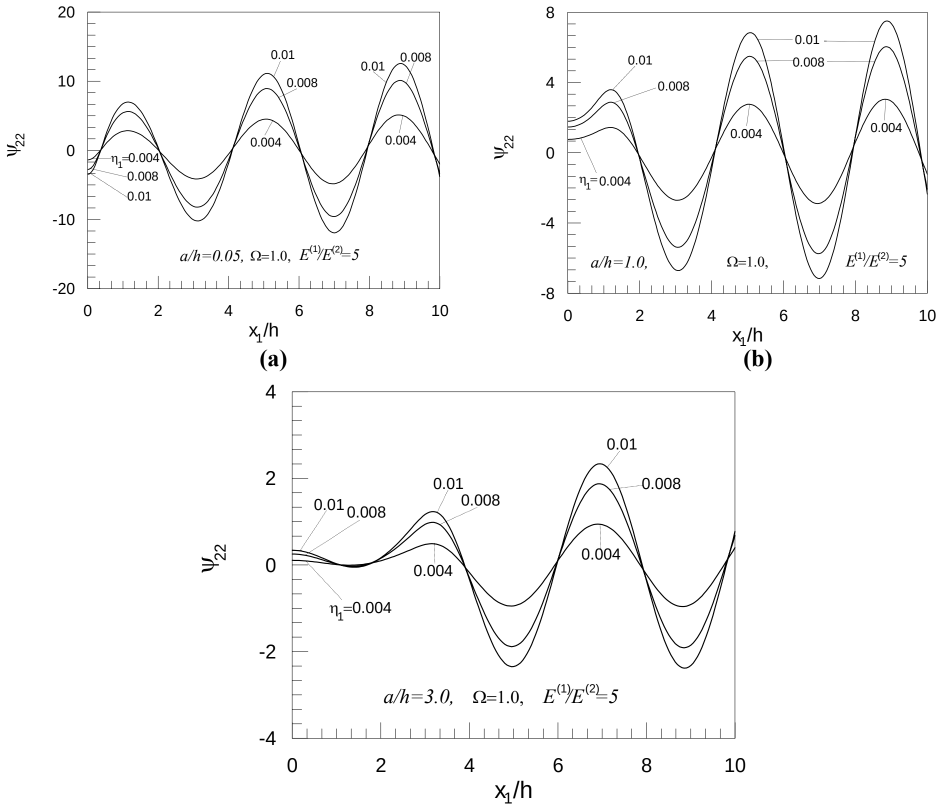


Figure 11. The distribution ψ_{22} with respect to x_1/h

In [28] the algorithm for obtaining numerical results was also developed and the concrete numerical results were presented for the pair of materials *Aluminium (layer) + Steel (half-plane)* as well as for *Steel (layer) + Aluminium (half-plane)*. These numerical results are involved in the normal stress $\sigma_{22}(x_1,0)h/P_0$. The analyses of the results are made in the viewpoint of the new context, namely in the context of the dependencies this stress and frequency of the external force and the influence of the pre-stretching of the covering layer on these dependencies. As an example, we here present the graphs (Fig. 12) of these dependencies for the pair of materials *Aluminium (layer) + Steel (half-plane)*. From these and other graphs constructed in [28] it follows that for both pairs of materials the dependence between $\sigma_{22}(x_1,0)h/P_0$ and

Ω is nonmonotonic. There is such a value of Ω for which $\sigma_{22}(x_1,0)h/P_0$ has its absolute maximum for the considered range of the change of Ω. These values of Ω and the corresponding absolute maximum values of $\sigma_{22}(x_1,0)h/P_0$ are called the "resonance" frequency and the "resonance" values of $\sigma_{22}(x_1,0)h/P_0$ respectively. The numerical results in [28] show that the values of the "resonance" frequency and the "resonance" values of $\sigma_{22}(x_1,0)h/P_0$ depend on the selected pair of materials (i.e. on the mechanical properties of the layer and half-space materials), on the inclination angle (i.e. on α) and on the initial tension of the covering layer.

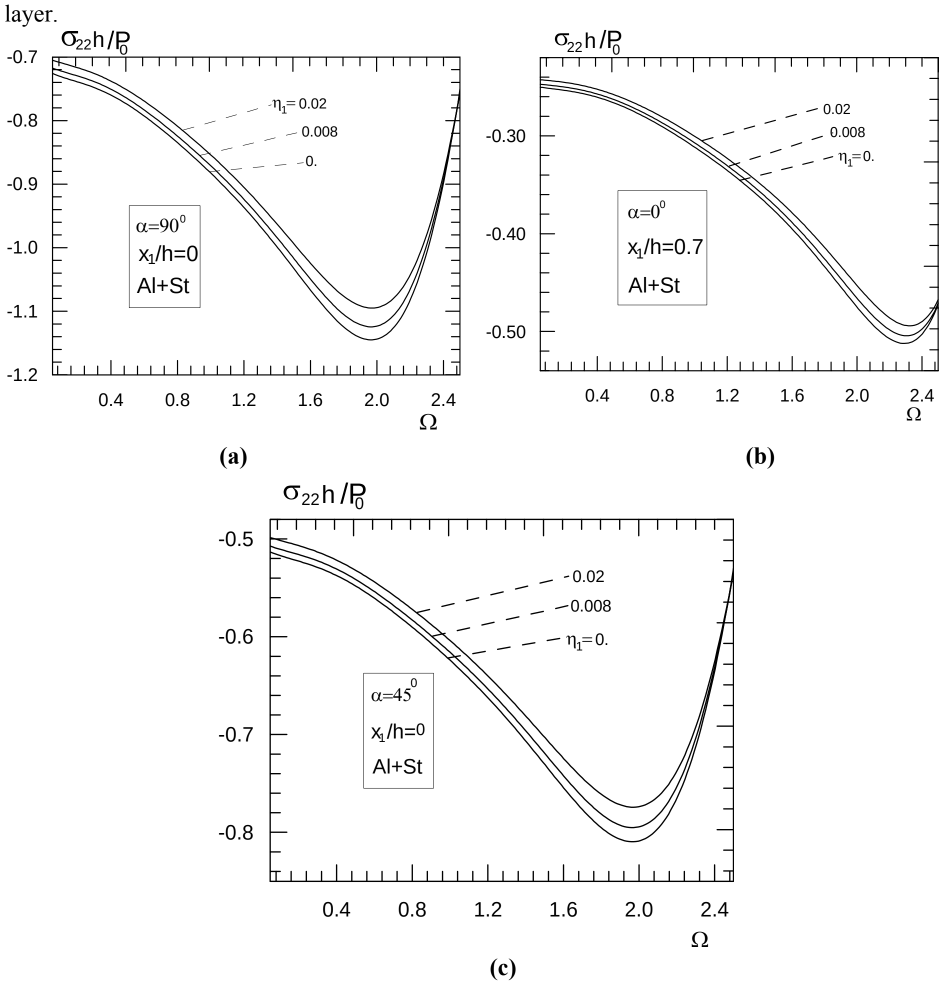


Figure 12. The graphs of the dependencies between $\sigma_{22}h/P_0$ and Ω

Note that the nonmonotonic character of the dependence between σ_{22} and Ω agrees with the well-known results presented in [22, 29 – 31] and others, according to which, the behaviour of the half-space under forced vibration is similar to that of the forced vibration of the system which comprises a mass, a parallel connected spring and a dashpot. It follows from the results obtained in [28] that the behaviour of the forced vibration of the half-plane covered by the layer is also similar to that of the mentioned system.

The other type of dynamical problems which are related to the above discussed ones regard the moving loading of a system comprises a pre-stretched covering layer and pre-strained half-plane. The first attempt in this field which was made by the use TLTEWISB in the plane-strain state had been carried out in [32]. Note that in [32] the dynamical response of the system consisting of the layer and pre-strained half-plane was considered. The equation of motion for the layer was described by the Timoshenko beam theory, but the equation of motion for the half plane was described by the TLTEWISB under finite initial strain state. The solution to the corresponding boundary-value problem is found by employing the exponential Fourier integral transformation. Concrete numerical investigations were made for the case where the constitutive relations for the half-plane material were given by the harmonic type of potential. Moreover, it was assumed that the speed of the moving load is constant and the subsonic case is considered. As a result of the numerical investigations the influence of the problem parameters on the critical velocity is studied. In the paper [33] the problem considered in [32] was studied by the use of complex potentials of the TLTEWISB [8].

In the paper [34] the investigations carried out in [32, 33] were developed for the case where the covering layer has also the initial strain and the equation of motion of this layer was also described by the TLTEWISB under SVSIDT. The considered problem is formulated by the equations (1), (2), (4), (5) (for the complete contact conditions), (6) (for incomplete contact conditions), (9) and (38) under the following boundary condition satisfied on the free face plane of the covering layer (Fig. 1).

$$\sigma_{12}^{(1)}\Big|_{x_2=h} = 0\,,\ \ \sigma_{22}^{(1)}\Big|_{x_2=h} = -P\delta(x_1 - Vt)\,, \tag{43}$$

where V is a constant and shows the loading velocity. It is assumed that

$$V < \min\left(c_2^{(1)}, c_2^{(2)}\right),\ c_2^{(m)} = \sqrt{\mu^{(m)}/\rho^{(m)}}\ . \tag{44}$$

The problem is solved by the use of the moving coordinate system $x'_2 = x_2$, $x'_1 = x_1 - Vt$ and the exponential Fourier transformation $f_F(s, x_2) = \int_{-\infty}^{+\infty} f(x'_1, x_2)e^{-isx'_1}dx'_1$. The Fourier transformation of the sought values can be expressed as follows:

$$\frac{1}{\det\|\alpha_{nm}(s)\|}\left(\det\|\beta_{nm}^{1}(s)\|, ..., \det\|\beta_{nm}^{k}(s), ...\|\right). \tag{45}$$

where $\alpha_{nm}(s)$ are the coefficients of the unknowns in the algebraic system of equations obtained from (43), (5) for the complete contact conditions or from (43), (6) for the incomplete contact conditions. Note that the expressions for $\|\beta_{nm}^{k}(s)\|$ are obtained from the $\|\alpha_{nm}(s)\|$ by replacing the k-th column in $\|\alpha_{nm}(s)\|$ with the right side of the above-noted algebraic equation system. Consequently, according to (45) the original of the sought values can be expressed as

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{\det\|\alpha_{nm}(s)\|}\left(\det\|\beta_{nm}^{1}(s)\|, ..., \det\|\beta_{nm}^{k}(s), ...\|e^{isx'_1}\right)ds\,. \tag{46}$$

It follows from (46) that the singular points of the integrated expression in (46) coincide with the roots of the equation

$$\det\|\alpha_{nm}(s)\| = 0 . \tag{47}$$

Consequently, the order of the singularity (denoted by r) of the integrated values coincide with the order of the roots of equation (47). It is known that in the case where $0 \le r < 1$ the integral can be calculated by the use of the well-known algorithm. In this case where $r = 1$ the calculation of the integral (46) is performed in the Cauchy's principal value sense. But in the case where $r > 1$ the integral does not have any meaning and under the velocity corresponding to this case the resonance type of phenomenon takes place. It is obvious that the critical velocity corresponds to the local minimum (or maximum) of the function $V = V(sh)$ which satisfy the equation (47). One of the main questions of the moving loading problems for layered materials in the subsonic state (44) is the determination of this critical velocity (denoted by V_{cr}) and investigation of the influence of the problem parameters on its values. The other question of the moving loading problem is the determination of the stress-strain in the considered mechanical system under $V < V_{cr}$. Namely these questions were analyzed in the paper [34] with the use of the parameters $\upsilon = V/c_2^{(1)}$, $\upsilon_{12} = \upsilon c_2^{(1)}/c_2^{(2)}$, $e = E^{(1)}/E^{(2)}$, $\eta_1 = \sigma_{11}^{(1)0}/E^{(1)}$, $\eta_2 = \sigma_{11}^{(2)0}/E^{(2)}$.

As an example, in Tables 3 and 4 some parts of the numerical results obtained in [34] for the case where $\rho^{(2)}/\rho^{(1)} = 0.5$, $\upsilon_{12} = \sqrt{0.5e}$ are given. Moreover, in [34] many other numerical results regarding $\upsilon_{cr} = V_{cr}/c_2^{(1)}$ and the stress distribution acting on the interface plane between the half-plane and covering layer are also presented.

Table 3. The values of the critical velocity $\upsilon_{cr} = V_{cr}/c_2^{(1)}$ for various η_2 under complete (upper numbers) and incomplete (lower numbers) contact conditions for the case $\eta_1 = 0.0$

e \ η_2	0.000	0.005	0.010	0.030
2	$\frac{0.8414}{0.7093}$	$\frac{0.8437}{0.7123}$	$\frac{0.8454}{0.7153}$	$\frac{0.8541}{0.7266}$
10	$\frac{0.4284}{0.3730}$	$\frac{0.4307}{0.3754}$	$\frac{0.4329}{0.3777}$	$\frac{0.4415}{0.3868}$

So far, in this section the 2D – two- dimensional (plane-strain state) problems has been reviewed. Now we consider the corresponding 3D – three – dimensional problems which have been investigated in the papers [35–38]. In these papers the time-harmonic three-dimensional (3D) Lamb's problem for the half-space covered bi-axially pre-stretched layer is considered. The investigations are carried out within the framework of the piecewise homogeneous body model by the use of the SVSIDT of the TLTEWISB and it is assumed that a time harmonic point located normal force acts on the free face plane of the covering layer. We here describe briefly the problem formulation and solution method used in [35–38].

Consider the half-space covered by a bi-axially pre-stretched layer. With the covering layer we associate a Lagrangian coordinate system $Ox_1x_2x_3$, which in the undeformed state, would coincide with a Cartesian coordinate system. Note that the covering layer and the half-space occupy the regions $\{-\infty < x_1 < +\infty , -h \le x_2 \le 0 , -\infty < x_3 < +\infty\}$ and

$\{-\infty < x_1 < +\infty\,,\ \ -\infty < x_2 \le -h\,,\ \ -\infty < x_3 < +\infty\}$ respectively (Fig.13). We assume that, before contact, the layer and the half-space are stressed separately in the directions of Ox_1 and Ox_3 axes, and homogeneous initial stress states appear in both materials.

Table 4. The values of the critical velocity $\upsilon_{cr} = V_{cr}/c_2^{(1)}$ for various η_2 under complete (upper numbers) and incomplete (lower numbers) contact conditions for the case $\eta_2 = 0.0$

e \ η_1	0.000	0.005	0.010	0.030
2	0.8414 / 0.7093	0.8468 / 0.7154	0.8521 / 0.7214	0.8723 / 0.7442
10	0.4284 / 0.3730	0.4319 / 0.3784	0.4348 / 0.3833	0.4358 / 0.3995

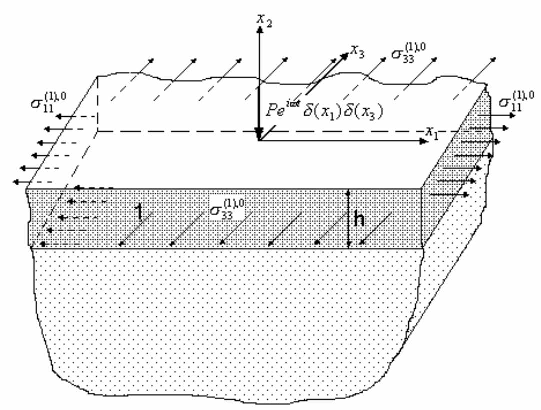


Figure 13. The geometry of the structure of the half-space covered by the layer

The values related to the layer and the half-space are denoted by upper indices (1) and (2), respectively. The values related to the initial stresses are denoted by upper indices (m),0 where m=1, 2. Moreover, below repeated indices will indicate summation over their ranges. However, underlined repeated indices are not to be summed.

The linearly elastic material of the layer and the half-space are to be taken as homogeneous and isotropic. The initial stresses in the layer and the half-space are determined within the framework of the classical linear theory of elasticity as follows

$$\sigma_{11}^{(m),0} = const_1.\,,\ \sigma_{33}^{(m),0} = const_3.\,,\ \sigma_{ij}^{(m),0} = 0 \ \text{ for } ij \ne 11;33 \tag{48}$$

For the considered case the equation of motion are

$$\frac{\partial \sigma_{ij}^{(\underline{m})}}{\partial x_j} + \sigma_{11}^{(\underline{m}),0}\frac{\partial^2 u_i^{(\underline{m})}}{\partial x_1^2} + \sigma_{33}^{(\underline{m}),0}\frac{\partial^2 u_i^{(\underline{m})}}{\partial x_3^2} = \rho_0^{(\underline{m})}\frac{\partial^2 u_i^{(\underline{m})}}{\partial t^2}\,,\ \ i;j = 1,2,3\,,\ \ m = 1,2. \tag{49}$$

For an isotropic compressible material one can write the following mechanical relations.

$$\sigma_{ij}^{(m)} = \lambda^{(m)}\theta^{(m)}\delta_{ij} + 2\mu^{(m)}\varepsilon_{ij}^{(m)}, \quad \theta^{(m)} = \varepsilon_{11}^{(m)} + \varepsilon_{22}^{(m)} + \varepsilon_{33}^{(m)}, \tag{50}$$

and the following geometrical relations

$$\varepsilon_{ij}^{(m)} = \frac{1}{2}\left(\frac{\partial u_i^{(m)}}{\partial x_j} + \frac{\partial u_j^{(m)}}{\partial x_i}\right). \tag{51}$$

It is assumed that the following complete contact conditions exist between the layer and the half-space.

$$u_i^{(1)}\Big|_{x_2=-h} = u_i^{(2)}\Big|_{x_2=-h}, \sigma_{i2}^{(1)}\Big|_{x_2=-h} = \sigma_{i2}^{(2)}\Big|_{x_2=-h}, i = 1, 2, 3. \tag{52}$$

In the free face plane of the covering layer, the following conditions are satisfied

$$\sigma_{32}^{(1)}\Big|_{x_2=0} = \sigma_{12}^{(1)}\Big|_{x_2=0} = 0, \ \sigma_{22}^{(1)}\Big|_{x_2=0} = -P_0\delta(x_1)\delta(x_3)e^{i\omega t} \tag{53}$$

In addition to these, it is also assumed that as $x_2 \to -\infty$ there is no reflection, which means all waves travel in the negative x_2 direction. In other words, we assume that

$$\left|u_i^{(2)}\right|, \left|\sigma_{ij}^{(2)}\right| < M = \text{constant, as } x_2 \to -\infty. \tag{54}$$

This completes the formulation of the problem. It should be noted that in the case where $\sigma_{11}^{(m),0} = 0, \sigma_{33}^{(m),0} = 0 \ (m = 1,2)$, the problem transforms to the corresponding formulation of the classical linear theory.

Now we stop briefly on the solution method of the above-formulated problem. Note that the method of solving the Lamb's problem has been developed intensively since Lamb [22]. Reference to various investigations is presented in papers [39, 40] and others. However, these developments have been made using the classical linear theory of elastic waves with homogeneous, isotropic or anisotropic half-spaces. Attention was focused on the construction of the integral expressions for the sought functions. Being attractive cases for numerical applications, these expressions were examined in some numerical study.

The afore-mentioned studies and those listed therein have a considerable significance in elastodynamics problems. These studies are based on various types of integral transformations with respect to space and time variables. For time harmonic excitations, the resulting multi-integrals were evaluated only in the far field by the method of stationary phase.

Recently, a semi-analytical FE technique called Spectral Finite Element Method (SFEM) has been developed in the papers [41-43] and others for investigation of elastodynamics problems in multilayered media. This method can also be applied to the investigation of the Lamb's problem for the multilayered half-space. However, until now, this method has only been developed for and applied to the investigation of the two-dimensional (spatial) elastodynamics problems. As in conventional semi-analytical FE methods, in SFEM, the sought expressions are represented in coordinates along the layer and time in series form. The kernels (i.e. the unknown coefficients of these series) depend on the coordinates changing through the thickness of the layer. These kernels are determined by employing one dimensional finite element modelling. In this case, the nodal displacements are related to the generalized nodal forces through the frequency and wavenumber dependent dynamic element stiffness matrix. The advantage of SFEM appears in cases where the considered body contains a large number of layers.

In the investigations considered in the present section the studying model is composed of a layer and half-space only. Moreover, the SFEM has not yet been developed for 3D time-harmonic Lamb's problem. Therefore in the reviewed investigations the integral transformation method was preferred. In connection with this in the papers [35-38] for the solution to the above formulated problem the double integral (Fourier) transformation method was employed. In this case the Lamé representations for displacements are used. These representations can be presented as follows.

$$\mathbf{u} = \nabla\phi + \nabla \times \boldsymbol{\psi}\,,\ \ \nabla.\boldsymbol{\psi} = 0 \tag{55}$$

where

$$\mathbf{u} = (u_1, u_2, u_3)\,,\ \ \boldsymbol{\psi} = (\psi_1, \psi_2, \psi_3). \tag{56}$$

In (55), the symbols $\times$ and $.$ show the vector and scalar products of vectors, respectively.

From Eqs. (55), (56) we can write

$$u_1 = \frac{\partial\phi}{\partial x_1} + \frac{\partial\psi_3}{\partial x_2} - \frac{\partial\psi_2}{\partial x_3}\,,\quad u_2 = \frac{\partial\phi}{\partial x_2} + \frac{\partial\psi_1}{\partial x_3} - \frac{\partial\psi_3}{\partial x_1}\,,\quad u_3 = \frac{\partial\phi}{\partial x_3} + \frac{\partial\psi_2}{\partial x_1} - \frac{\partial\psi_1}{\partial x_2}\,. \tag{57}$$

After some manipulation, the following equations for the functions ϕ, ψ_1, ψ_2 and ψ_3 from equations (49)-(51):

$$\nabla^2\phi^{(\underline{m})} + \frac{\sigma_{11}^{(\underline{m}),0}}{\lambda^{(\underline{m})} + 2\mu^{(\underline{m})}}\frac{\partial^2\phi^{(\underline{m})}}{\partial x_1^2} + \frac{\sigma_{33}^{(\underline{m}),0}}{\lambda^{(\underline{m})} + 2\mu^{(\underline{m})}}\frac{\partial^2\phi^{(\underline{m})}}{\partial x_3^2} = \frac{1}{\left(c_1^{(\underline{m})}\right)^2}\frac{\partial^2\phi^{(\underline{m})}}{\partial t^2}\,,$$

$$\nabla^2\psi_i^{(\underline{m})} + \frac{\sigma_{11}^{(\underline{m}),0}}{\mu^{(\underline{m})}}\frac{\partial^2\psi_i^{(\underline{m})}}{\partial x_1^2} + \frac{\sigma_{33}^{(\underline{m}),0}}{\mu^{(\underline{m})}}\frac{\partial^2\psi_i^{(\underline{m})}}{\partial x_3^2} = \frac{1}{\left(c_2^{(\underline{m})}\right)^2}\frac{\partial^2\psi_i^{(\underline{m})}}{\partial t^2}\,,$$

$$\frac{\partial\psi_1^{(m)}}{\partial x_1} + \frac{\partial\psi_2^{(m)}}{\partial x_2} + \frac{\partial\psi_3^{(m)}}{\partial x_3} = 0\,, \tag{58}$$

where $c_1^{(\underline{m})} = \sqrt{\left(\lambda^{(\underline{m})} + 2\mu^{(\underline{m})}\right)/\rho_0^{(\underline{m})}}$ is the speed of the dilatation wave. Under the conditions $\sigma_{11}^{(m),0} = 0$ and $\sigma_{33}^{(m),0} = 0$, the equation (58) coincide with the corresponding ones derived in the classical linear theory of elastodynamics [16].

By replacing, $\partial^2/\partial t^2$ with $-\omega^2$, the same equations and conditions for the amplitude of the sought quantities are obtained. For the solution of these equations the double Fourier transformation

$$f_{13F}(s_1, x_2, s_3) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x_1, x_2, x_3)e^{-i(s_1x_1+s_3x_3)}dx_1dx_3 \tag{58}$$

with respect to the coordinates x_1 and x_3 is employed.

The original unknowns that were sought can now be represented as

$$\left\{u_{\underline{n}}^{(\underline{m})}, \sigma_{\underline{n}\underline{j}}^{(\underline{m})}, \varepsilon_{\underline{n}\underline{j}}^{(\underline{m})}\right\} = \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{u_{\underline{n}13F}^{(\underline{m})}, \sigma_{\underline{n}\underline{j}13F}^{(\underline{m})}, \varepsilon_{\underline{n}\underline{j}13F}^{(\underline{m})}\right\}e^{i(s_1x_1+s_3x_3)}ds_1ds_3 \tag{59}$$

It should be noted that in the investigations [35-38] the main difficulties arise under calculation of the integrals (59). For this purpose the algorithm used under studying the corresponding two-dimensional problems is developed for the considered three-dimensional problems. Taking the high significance of this algorithm in these investigations into account we here briefly consider its principal moments.

To simplify the matters, we consider the calculation of the integral for $\sigma_{22}^{(m)}$, that is the integral

$$\sigma_{22}^{(m)} = \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\sigma_{2213F}^{(m)}(s_1, x_2, s_3)e^{i(s_1x_1+s_3x_3)}ds_1ds_3 \ . \tag{60}$$

Introduce the following notation

$$\varphi(x_1, x_2, x_3) = \sigma_{22}^{(m)}(x_1, x_2, x_3)\ ,\ \ \varphi_{13F}(s_1, x_2, s_3) = \sigma_{2213F}^{(m)}(s_1, x_2, s_3)\ ,$$

$$\varphi_{3F}(x_1, x_2, s_3) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\varphi_{13F}(s_1, x_2, s_3)e^{is_1x_1}ds_1 \tag{61}$$

Using symmetry and Eq. (61), the integral (60) can be represented as follows:

$$\varphi(x_1, x_2, x_3) = \frac{1}{\pi^2}\int_0^{\infty}\int_0^{\infty}\varphi_{13F}(s_1, x_2, s_3)\cos s_1x_1 \cos s_3x_3ds_1ds_3 \tag{62}$$

The following explains how the reduced integral (62) is calculated. First, the integral (62) is replaced by a corresponding definite integral, by using the following approximation

$$\int_0^{+\infty}\int_0^{+\infty}\varphi_{13F}(s_1, x_2, s_3)\cos s_1x_1 \cos s_3x_3ds_1ds_3 \approx \int_0^{s_{3*}}\int_0^{s_{1*}}\varphi_{13F}(s_1, x_2, s_3)\cos s_1x_1 \cos s_3x_3ds_1ds_3 \tag{63}$$

The values of S_{1*} and S_{3*} in Eq. (63) are defined from the convergence requirement.

For the calculation of the definite integral in Eq. (63), first, the interval $[0, S_{3*}]$ is subdivided into shorter intervals $[S_{3i}, S_{3i+1}]$, i=0,1,2,...,*N*, $S_{30} = 0$, $S_{3N} = S_{3*}$, where $\bigcup_{i=0}^{N}[S_{3i}, S_{3i+1}] = [0, S_{3*}]$ and $\bigcap_{i=0}^{N}(S_{3i}, S_{3i+1}) = \varnothing$. Then, the definite integral becomes

$$\int_0^{S_{3*}}\int_0^{S_{1*}}(.)ds_1ds_3 = \sum_{i=0}^{N}\int_{S_{3i}}^{S_{3i+1}}\left(\int_0^{S_{1*}}(.)ds_1\right)ds_3\ , \tag{64}$$

where $(.)$ denotes the integrand.

Consequently, we obtain from Eqs. (64) and (62) that

$$\int_0^{S_{3*}}\int_0^{S_{1*}}(.)ds_1ds_3 = \sum_{i=0}^{N}\int_{S_{3i}}^{S_{3i+1}}\varphi_{3F}(x_1, x_2, s_3)\cos s_3x_3ds_3 \ . \tag{65}$$

For calculations of the integrals (65) in the intervals $[S_{3i}, S_{3i+1}]$, we use the Gauss integration algorithm, which is necessary to find the values of $\varphi_{3F}(x_1, x_2, s_3)$ at certain nodal points $s_3 = s_{3k}^{/}$. Thus, the calculation of the integral (64) is reduced to the calculation of the integral

$$\varphi_{3F}(x_1, x_2, s_{3k}^{/}) = \int_0^{S_{1*}}\varphi_{13F}(s_1, x_2, s_{3k}^{/})\cos s_1x_1ds_1 \tag{66}$$

Now we consider the calculation of the integral (66). Numerical investigations show that the function $\varphi_{13F}(s_1, x_2, s_{3k}^{/})$ has a singular point for each selected $s_{3k}^{/}$ and the location of these singular points in the intervals $[0, S_{1*}]$ depends on $s_{3k}^{/}$.

It should be noted that, as in the two-dimensional problems analyzed above, the expressions of the Fourier transformations of the sought values contain the unknowns which are determined from the contact (52) and boundary conditions (53), (54). In this case the closed linear system of algebraic equations with respect to these unknowns are obtained from the (52)-(54), according to which, the unknowns can be expressed as follows.

$$\frac{1}{\det\left\|\alpha_{ij}(s_1, s_{3k}^{/})\right\|}\left(\det\left\|\beta_{ij}^{1}\right\|; \det\left\|\beta_{ij}^{2}\right\|; \ldots; \det\left\|\beta_{ij}^{k}\right\|; \ldots\right)(s_1, s_{3k}^{/}) \quad . \tag{67}$$

Consequently, the afore-mentioned singular points coincide with the roots of the equation

$$\det\left\|\alpha_{ij}(s_1, s_{3k}^{/})\right\| = 0, \; i; j = 1, 2, 3, ..., 9 \tag{68}$$

in s_1, where $\alpha_{ij}(s_1, s_{3k}^{/})$ are the coefficients of the unknowns in the afore-mentioned algebraic equation system. Note that the expressions for $\left\|\beta_{ij}^{1}\right\|, \ldots, \left\|\beta_{ij}^{k}\right\|$ are obtained from $\left\|\alpha_{ij}(s_1, s_{3k}^{/})\right\|$ by replacing the corresponding column of $\left\|\alpha_{ij}(s_1, s_{3k}^{/})\right\|$ with the right side of the algebraic equation system.

A numerical analysis shows that the order of the roots of the equation (68) is one. Therefore, the order of all singular points is also one. Taking this situation into account in the solution to equation (68), we employ the well-known bisection method.

Let us denote the roots of the equation (68) as

$$s_{11}(s_{3k}^{/}) < s_{12}(s_{3k}^{/}) < \ldots < s_{1k}(s_{3k}^{/}) < \ldots < s_{1M}(s_{3k}^{/}) \; . \tag{69}$$

The number M in (69) depends mainly on the values of $s_{3k}^{/}$, the dimensionless frequency $\Omega = \omega h / c_2^{(1)}$, and the mechanical and geometrical parameters of the layer and half-space.

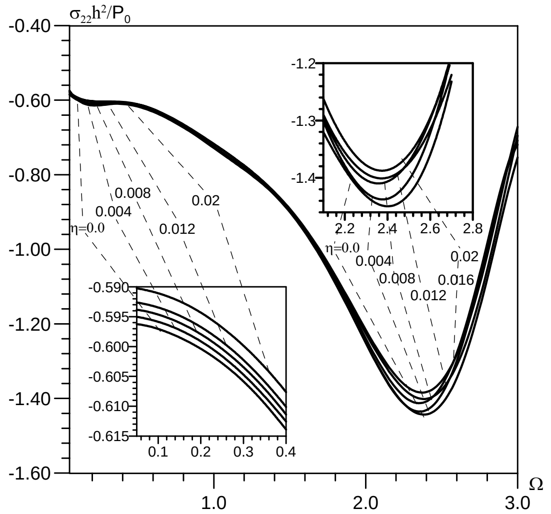
After determining the roots (69), the interval of integration $[0, S_{1*}]$ in (29) is partitioned as follows

$$\int_0^{S_{1*}} (.)ds_1 = \int_0^{S_{11}(S_{3k}^{/})-\varepsilon} (.)ds_1 + \int_{S_{11}(S_{3k}^{/})+\varepsilon}^{S_{12}(S_{3k}^{/})-\varepsilon} (.)ds_1 + \cdots + \int_{S_{1M}(S_{3k}^{/})+\varepsilon}^{S_{1*}} (.)ds_1 \tag{70}$$

Then, the calculation of the integral (70) is performed in the Cauchy's principal value sense. Here ε is a very small value determined numerically from the convergence requirement of the integral (70). Each interval $\left[s_{1n}(s_{3k}^{/}) + \varepsilon, \, s_{1n+1}(s_{3k}^{/}) - \varepsilon\right]$ is further divided into a certain number of shorter intervals, which are used in Gauss integration algorithm. All these procedures are performed using the programmes written in C++. Note that all numerical investigations carried out in the papers [35-38] were made by the use of the foregoing algorithm. In the paper [35] these investigations were made for the case where $\sigma_{11}^{(1),0} > 0$, $\sigma_{33}^{(1)} > 0$, $\sigma_{11}^{(2),0} =$

$\sigma_{33}^{(2),0} = 0$, $\nu^{(1)} = \nu^{(2)} = 0.3$, $E^{(1)}/E^{(2)} = 2$ and $\rho_0^{(1)} = \rho_0^{(2)}$. At the same time, the parameters $\eta_{11} = \sigma_{11}^{(1),0}/\mu^{(1)}$, $\eta_{13} = \sigma_{33}^{(1),0}/\mu^{(1)}$ were introduced and the distribution of the stresses $\sigma_{i2} = \sigma_{i2}^{(1)}\big|_{x_2=-h} = \sigma_{i2}^{(2)}\big|_{x_2=-h}$ $(i = 1,2)$ with respect to x_1/h was analyzed. However, in the papers [36,37] the numerical results are obtained for the following two materials: Rubber (shortly Rb) with properties $\rho_0 = 0.93x10^{-3}\ kg/m^3$, $\nu = 0.49958$, $c_1 = 1040\ m/s$, $c_2 = 30\ m/s$; Aluminium (shortly Al) with properties $\rho_0 = 2.7x10^{-3}\ kg/m^3$, $\nu = 0.35$, $c_1 = 6420\ m/s$, $c_2 = 3110\ m/s$, where ρ_0, ν, c_1 and c_2 denote the density, Poisson's ratio, the speed of dilatation and distortion waves, respectively. The numerical investigation is carried out for the following two cases: Case I: (Rb+Al), Layer=Rubber, Half-space = Aluminium, Case II: (AL+Rb), Layer=Aluminium, Half - space=Rubber. In [37] the analysis was made for the stress $\sigma_{22}(x_1,x_3) = \sigma_{22}^{(1)}(x_1,-h,x_3) = \sigma_{22}^{(2)}(x_1,-h,x_3)$. It is assumed that $\sigma_{11}^{(1),0} > 0$, $\sigma_{11}^{(2),0} = \sigma_{33}^{(1),0} = \sigma_{33}^{(2),0} = 0$, $0 < \Omega \le 2.0$ and the influence of the parameters η_{11} and Ω the distribution of the stress $\sigma_{22}(x_1,x_3)$ with respect to x_1/h and x_3/h is studied. In the paper [37] this study is developed for the other stresses acting on the interface plane and for the stresses $\sigma_{11}(x_1,x_3) = \sigma_{11}^{(1)}(x_1,-h,x_3)$, $\sigma_{33}(x_1,x_3) = \sigma_{33}^{(1)}(x_1,-h,x_3)$.

In the paper [38] the main attention is focused on the dependencies between $\sigma_{22}(0,0)$ and Ω for the pair of materials Aluminium (layer) + Steel (half-space) and Steel (layer) + Aluminium (half-space). The graphs of these dependencies are given in Figs. 14 and 15 respectively.



Figüre 14. The graphs of the dependencies between $\sigma_{22}h^2/P_0$ and Ω for Aluminium (layer) + Steel (half-space).

It follows from these graphs that the analyzed dependencies are nonmonotonic. Consequently, as in the plane-strain state, the dynamical behaviour of the stratified half-space is similar to that of the system composed by a mass, a spring and a dashpot. Moreover, these graphs show that with η_{11} the "resonance" values of the σ_{22} decrease, but the "resonance" frequency $\Omega = \Omega_*$ increase (decrease) for the pair of materials St + Al (Al + St).

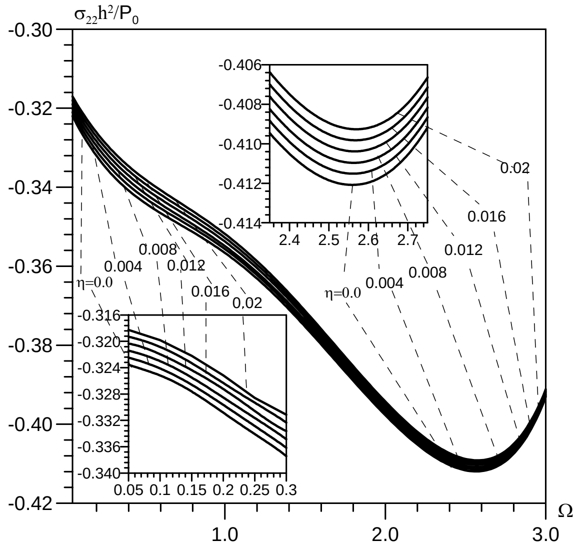


Figure 15. The graphs of the dependencies between $\sigma_{22}h^2/P_0$ and Ω for Steel (layer) + Aluminium (half-space)

All investigations considered above and regarding the time-harmonic stress state analyses were made within the framework of the following two basic assumptions:

1) the initial strain state is determined by employing of the linear theory of elasticity;
2) the materials of the layer and half-space are the linear elastic ones.

It is known that the mechanical effects caused by the initial stresses (see, for example, Ref. [5,9]) in the qualitative and in the quantitative sense depend significantly on the values of the third order elastic constants. This situation requires the modelling of the stress-strain law of the layer and half-space materials by the nonlinear mechanical relations. The investigations reviewed and considered in [4-9] and in many others show that for this purpose it is suitable to use the Murnaghan potential [13] and to refuse from the foregoing first assumption. Moreover, it is also necessary to refuse from the foregoing second assumptions under consideration of the rubber-like materials which are employed widely in the key industries.

Taking the above discussions into account in the papers [44 - 46] the axisymmetric time-harmonic Lamb's problem for the system composed by the pre-strained layer and half-space are studied without the foregoing two basic restrictions. Instead of these restrictions in the paper [44] it is supposed that: 1) the initial strain state in the layer and half-space is determined by employing geometrical nonlinear theory of elasticity; 2) the linearized mechanical relations of the components are described by the Murnaghan potential. In this case the problem formulation is written through the equations (22)-(29) and it is assumed that the parameters $\lambda_i^{(k)}$ in (22) are determined as follows.

$$\left(\lambda_1^{(\underline{k})}\right)^2=\left(\lambda_2^{(\underline{k})}\right)^2=1+\left(\frac{\lambda^{(\underline{k})}}{\mu^{(\underline{k})}}+2\right)\frac{\sigma^{(\underline{k}),0}}{\mu^{(\underline{k})}}\frac{1}{\left(3\lambda^{(\underline{k})}/\mu^{(\underline{k})}+2\right)},$$

$$\left(\lambda_3^{(\underline{k})}\right)^2=1-\frac{\lambda^{(\underline{k})}}{\mu^{(\underline{k})}}\frac{\sigma^{(\underline{k}),0}}{\mu^{(\underline{k})}}\frac{1}{\left(3\lambda^{(\underline{k})}/\mu^{(\underline{k})}+2\right)}, \tag{71}$$

where $\sigma^{(k),0}=\sigma_{rr}^{(k),0}=\sigma_{\theta\theta}^{(k),0}$. Moreover, the components $\omega'^{(k)}_{ij\alpha\beta}$ in (25) in the paper [44] are determined by the following expressions.

$$\omega'^{(k)}_{1111}=\omega'^{(k)}_{2222}=\omega'^{(k)}_{2211}=\omega'^{(k)}_{1122}=\left(\lambda_3^{(k)}\right)^{-1}\left(\left(\lambda_1^{(k)}\right)^2A_{11}^{(k)}+\sigma^{(k),0}\right),\ \omega'^{(k)}_{3333}=\left(\lambda_3^{(k)}\right)^2\left(\lambda_1^{(k)}\right)^{-2}A_{33}^{(k)},$$

$$\omega'^{(k)}_{3311}=\omega'^{(k)}_{1133}=\omega'^{(k)}_{2233}=\lambda_3^{(k)}A_{13}^{(k)},\ \omega'^{(k)}_{1331}=\left(\lambda_3^{(k)}\right)^{-1}\left(\left(\lambda_3^{(k)}\right)^2\mu_{13}^{(k)}+\sigma^{(k),0}\right),$$

$$\omega'^{(k)}_{1313}=\omega'^{(k)}_{3131}=\lambda_3^{(k)}\mu_{13}^{(k)},\ \omega'^{(k)}_{3113}=\lambda_3^{(k)}\mu_{13}^{(k)}, \tag{72}$$

where

$$A_{11}^{(k)}=\left(\lambda^{(k)}+2\mu^{(k)}\right)\left(1+\frac{\sigma^{(k),0}}{\left(\lambda^{(k)}+2\mu^{(k)}\right)3K_0^{(k)}}\left(4a^{(k)}+\frac{\lambda^{(k)}+4\mu^{(k)}}{\mu^{(k)}}2b^{(k)}+\frac{\lambda^{(k)}+2\mu^{(k)}}{\mu^{(k)}}c^{(k)}\right)\right)$$

,

$$A_{33}^{(k)}=\left(\lambda^{(k)}+2\mu^{(k)}\right)\left(1+\frac{\sigma^{(k),0}}{\left(\lambda^{(k)}+2\mu^{(k)}\right)3K_0^{(k)}}\left(4a^{(k)}-\frac{\lambda^{(k)}-\mu^{(k)}}{\mu^{(k)}}4b^{(k)}+\frac{\lambda^{(k)}}{\mu^{(k)}}2c^{(k)}\right)\right),$$

$$A_{13}^{(k)}=\lambda^{(k)}\left(1+\frac{\sigma^{(k),0}}{3K_0^{(k)}\lambda^{(k)}}(4a^{(k)}-\frac{\lambda^{(k)}-2\mu^{(k)}}{\mu^{(k)}}b^{(k)}\right)\Bigg),\ \mu_{13}^{(k)}=\mu\left(1+\frac{\sigma^{(k),0}}{3K_0^{(k)}\mu^{(k)}}\times\right.$$

$$\left.\left(2b^{(k)}-\frac{\lambda^{(k)}-2\mu^{(k)}}{\mu^{(k)}}\frac{c^{(k)}}{4}\right)\right),K_0^{(k)}=\lambda^{(k)}+\frac{2}{3}\mu^{(k)} \tag{73}$$

The following boundary and contact conditions are satisfied

$$T'^{(1)}_{z'z'}\Big|_{z'=0}=-P_0\delta(r')e^{i\omega t},\ T'^{(1)}_{z'r'}\Big|_{z'=0}=0,\ T'^{(1)}_{z'z'}\Big|_{z'=-h'}=T'^{(2)}_{z'z'}\Big|_{z'=-h'},T'^{(1)}_{z'r'}\Big|_{z'=-h'}=T'^{(2)}_{z'r'}\Big|_{z'=-h'},$$

$$u'^{(1)}_{r'}\Big|_{z'=-h'}=u'^{(2)}_{r'}\Big|_{z'=-h'},\ u'^{(1)}_{z'}\Big|_{z'=-h'}=u'^{(2)}_{z'}\Big|_{z'=-h'},$$

$$\left|u'^{(2)}_{r'}\right|,\ \left|u'^{(2)}_{z'}\right|,\ \left|\sigma'^{(2)}_{r'r'}\right|,\ \left|\sigma'^{(2)}_{\theta'\theta'}\right|,\ \left|\sigma'^{(2)}_{z'z'}\right|,\ \left|\sigma'^{(2)}_{r'z'}\right|<M=\text{const}\ \text{ as }\ z'\to-\infty \tag{74}$$

Note that in (74) the index z' and the notation z' are used instead of the index 3 and notation y'_3 respectively in (24) and (25).

For the solution to the considered problem, according to [7,9], the following representation for displacements is used.

$$u'^{(k)}_{r'}=-\frac{\partial^2}{\partial r'\partial z'}X^{(k)},\ u'^{(\underline{k})}_{z'}=\frac{1}{\omega'^{(\underline{k})}_{1133}+\omega'^{(\underline{k})}_{1313}}\left(\omega'^{(\underline{k})}_{1111}\Delta'_1+\omega'^{(\underline{k})}_{3113}\frac{\partial^2}{\partial z'^2}-\rho'^{(\underline{k})}\frac{\partial^2}{\partial t^2}\right)X^{(\underline{k})}, \tag{75}$$

where $X^{(k)}$ satisfies the following equation.

$$\left[\left(\Delta'+\left(\xi'^{(\underline{k})}_{2}\right)^2\frac{\partial^2}{\partial z'^2}\right)\left(\Delta'_1+\left(\xi'^{(\underline{k})}_{3}\right)^2\frac{\partial^2}{\partial z'^2}\right)-\rho'^{(\underline{k})}\left(\frac{\omega'^{(\underline{k})}_{1111}+\omega'^{(\underline{k})}_{1331}}{\omega'^{(\underline{k})}_{1111}\,\omega'^{(\underline{k})}_{1331}}\Delta'_1+\right.\right.$$

$$\left.\left.\frac{\omega'^{(\underline{k})}_{3333}+\omega'^{(\underline{k})}_{3113}}{\omega'^{(\underline{k})}_{1111}\,\omega'^{(\underline{k})}_{1331}}\frac{\partial^2}{\partial z'^2}\right)\frac{\partial^2}{\partial t^2}+\frac{\left(\rho'^{(\underline{k})}\right)^2}{\omega'^{(\underline{k})}_{1111}\,\omega'^{(\underline{k})}_{1331}}\frac{\partial^4}{\partial t^4}\right]X'^{(\underline{k})}=0\,. \tag{76}$$

In (75) and (76) the following notation is used.

$$\Delta'_1=\frac{d^2}{dr'^2}+\frac{1}{r'}\frac{d}{dr'},\quad \left(\xi'^{(\underline{k})}_{2,3}\right)^2=d^{(\underline{k})}\pm\left[\left(d^{(\underline{k})}\right)^2-\omega'^{(\underline{k})}_{3333}\,\omega'^{(\underline{k})}_{3113}\left(\omega'^{(\underline{k})}_{1111}\,\omega'^{(\underline{k})}_{1331}\right)^{-1}\right]^{\frac{1}{2}},$$

$$d^{(\underline{k})}=\left(2\omega'^{(\underline{k})}_{1111}\,\omega'^{(\underline{k})}_{1331}\right)^{-1}\left[\omega'^{(\underline{k})}_{1111}\,\omega'^{(\underline{k})}_{3333}+\omega'^{(\underline{k})}_{1331}\,\omega'^{(\underline{k})}_{3113}-\left(\omega'^{(\underline{k})}_{1133}+\omega'^{(\underline{k})}_{1313}\right)^2\right]. \tag{77}$$

By replacing $\partial^2/\partial t^2$, $\partial^4/\partial t^4$ with $-\omega^2$ and ω^4 respectively the same equations and conditions are obtained for the amplitude of the sought quantities. For the solution to these equations the Hankel integral representation is used for the function $X'^{(k)}$:

$$X'^{(\underline{k})}=\int_0^\infty F^{(\underline{k})}(s)e^{\gamma^{(k)}z'}J_0(sr')sds\,, \tag{78}$$

where $J_0(x)$ is the Bessel function with zeroth order. By the employing of the above–detailed algorithm the numerical results are obtained for the concrete selected materials. As an example, here we consider the dependencies between $\sigma_{zz}h^2/P_0$ and Ω for the pairs Steel (layer) + Al (half-space) (Fig. 16) and Acrylic Plastic (layer) + Al (half-space) (Fig.17).

In the papers [45, 46] the investigations carried out in the paper [44] was developed for the case where the initial strain state in the components of the considered system is a finite one. It is assumed that the material of the layer and half-space are incompressible and elastic relations for those are given through the Treloar's potential and the initial strain state in the components is

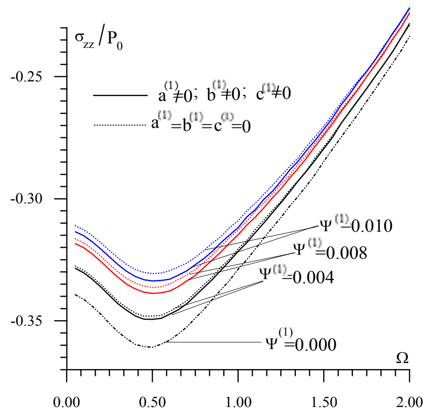


Figure 16. The dependencies between $\sigma_{zz}h^2/P_0$ and Ω for the pairs Steel (layer) + Al (half-space)

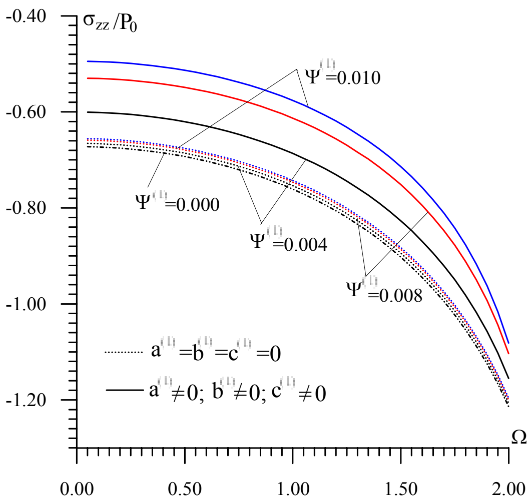


Figure 17. The dependencies between $\sigma_{zz}h^2/P_0$ and Ω for the pairs Acrylic Plastic (layer) + Al (half-space)

determined as (22). In this case the incompressibility condition in the initial state taking through the relation $\lambda_1^{(k)}\lambda_2^{(k)}\lambda_3^{(k)}=1$ $(k=1,2)$ into account the notation $\lambda_1^{(k)}=\lambda_2^{(k)}=\lambda^{(k)}$, $\lambda_3^{(k)}=(\lambda^{(k)})^{-2}$ is introduced. Thus, the mathematical formulation of the problems considered in [45, 46] is reduced to the following boundary value problem.

The equations of motion are

$$\frac{\partial}{\partial r'}Q'^{(k)}_{r'r'}+\frac{\partial}{\partial y'_3}Q'^{(k)}_{r'3}+\frac{1}{r'}\left(Q'^{(k)}_{r'r'}-Q'^{(k)}_{\theta'\theta'}\right)=\rho'^{(k)}\frac{\partial^2}{\partial t^2}u'^{(k)}_{r'},$$

$$\frac{\partial}{\partial r'}Q'^{(k)}_{3r'}+\frac{\partial}{\partial y'_3}Q'^{(k)}_{33}+\frac{1}{r'}Q'^{(k)}_{3r'}=\rho'^{(k)}\frac{\partial^2}{\partial t^2}u'^{(k)}_{3}. \tag{79}$$

The mechanical relations are

$$Q'^{(k)}_{r'r'}=\chi'^{(k)}_{1111}\frac{\partial u'^{(k)}_{r'}}{\partial r'}+\chi'^{(k)}_{1122}\frac{u'^{(k)}_{r'}}{r'}+\chi'^{(k)}_{1133}\frac{\partial u'^{(k)}_{3}}{\partial y'_3}+p'^{(k)},$$

$$Q'^{(k)}_{\theta'\theta'}=\chi'^{(k)}_{2211}\frac{\partial u'^{(k)}_{r'}}{\partial r'}+\chi'^{(k)}_{2222}\frac{u'^{(k)}_{r'}}{r'}+\chi'^{(k)}_{2233}\frac{\partial u'^{(k)}_{3}}{\partial y'_3}+p'^{(k)},$$

$$Q'^{(k)}_{33}=\chi'^{(k)}_{3311}\frac{\partial u'^{(k)}_{r'}}{\partial r'}+\chi'^{(k)}_{3322}\frac{u'^{(k)}_{r'}}{r'}+\chi'^{(k)}_{3333}\frac{\partial u'^{(k)}_{3}}{\partial y'_3}+p'^{(k)},$$

$$Q'^{(k)}_{r'3}=\chi'^{(k)}_{1313}\frac{\partial u'^{(k)}_{r'}}{\partial y'_3}+\chi'^{(k)}_{1331}\frac{\partial u'^{(k)}_{3}}{\partial r'},\quad Q'^{(k)}_{3r'}=\chi'^{(k)}_{3113}\frac{\partial u'^{(k)}_{r'}}{\partial y'_3}+\chi'^{(k)}_{3131}\frac{\partial u'^{(k)}_{3}}{\partial r'}. \tag{80}$$

In (79) and (80) through the $Q'^{(k)}_{r'r'},\ldots,\ Q'^{(k)}_{3r'}$ the perturbations of the components of Kirchhoff stress tensor are denoted. The notation $p'^{(k)} = p'^{(k)}(r', y'_3, t)$ is an unknown function (a Lagrange multiplier). The constants $\chi'^{(k)}_{1111},\ldots,\chi'^{(k)}_{3333}$ in (79), (80) are determined through the mechanical constants of the layer and half-space materials and through the initial stress state.

As it has been noted above it is assumed that the elasticity relations of the layer and half-space materials are given by Neo-Hooken type (Treloar's) potential. This potential is written as follows:

$$\Phi = C_{10}(I_1 - 3), \quad I_1 = 3 + 2A_1,\ A_1 = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{33}, \tag{81}$$

where C_{10} is an elastic constant; A_1 is the first algebraic invariant of the Green's strain tensor, ε_{rr}, $\varepsilon_{\theta\theta}$ and ε_{33} are the components of this tensor. For the considered axisymmetric case the components of the Green's strain tensor are determined through the components of the displacement vector by the expressions (28). In this case the components S_{ij} of the Lagrange stress tensor are determined as follows:

$$S_{rr} = \frac{\partial\Phi}{\partial\varepsilon_{rr}} + pg^*_{rr},\ S_{\theta\theta} = \frac{\partial\Phi}{\partial\varepsilon_{\theta\theta}} + pg^*_{\theta\theta},\ S_{33} = \frac{\partial\Phi}{\partial\varepsilon_{33}} + pg^*_{33}\ \ S_{r3} = \frac{\partial\Phi}{\partial\varepsilon_{r3}},\ S_{r3} = S_{3r},$$

$$g^*_{rr} = 1 + 2\frac{\partial u_r}{\partial r} + \left(\frac{\partial u_r}{\partial r}\right)^2 + \left(\frac{\partial u_3}{\partial r}\right)^2,\ g^*_{33} = 1 + 2\frac{\partial u_3}{\partial y_3} + \left(\frac{\partial u_3}{\partial y_3}\right)^2 + \left(\frac{\partial u_r}{\partial y_3}\right)^2,$$

$$g^*_{\theta\theta} = 1 + 2\frac{u_r}{r} + \frac{1}{r^2}\left(\frac{u_r}{r}\right)^2. \tag{82}$$

Note that the expressions (81), (28) and (82) are written in the arbitrary system of cylindrical coordinate system without any restriction related to the associated of this system to the natural or initial state of the considered layer and half-space.

For the considered case the relations between the perturbation of the Kirchoff stress tensor and the perturbation of the components of the Lagrange stress tensor can be written as follows:

$$Q'^{(k)}_{r'r'} = \lambda^{(\underline{k})}S^{(\underline{k})}_{r'r'} + S^{0(\underline{k})}_{rr}\frac{\partial u'^{(\underline{k})}_{r'}}{\partial r'},\ Q'^{(k)}_{\theta'\theta'} = \lambda^{(\underline{k})}S^{(\underline{k})}_{\theta'\theta'} + S^{0(\underline{k})}_{rr}\frac{u'^{(\underline{k})}_{r'}}{r'},\ \ Q'^{(k)}_{33} = \left(\lambda^{(\underline{k})}\right)^{-2}S^{(\underline{k})}_{33},$$

$$Q'^{(k)}_{r'3} = \lambda^{(\underline{k})}S^{(\underline{k})}_{r'3} + S^{0(\underline{k})}_{rr}\frac{\partial u'^{(\underline{k})}_{3}}{\partial r'},\quad Q'^{(k)}_{3r'} = \left(\lambda^{(\underline{k})}\right)^{-2}S^{(\underline{k})}_{3r'}. \tag{83}$$

It should be noted that the incompressibility condition of the layer and half-space materials must be added to the above equations. This condition for the considered case can be written as follows:

$$\frac{\partial u'^{(\underline{k})}_{r'}}{\partial r'} + \frac{u'^{(\underline{k})}_{r'}}{r'} + \frac{\partial u'^{(\underline{k})}_{3}}{\partial y'_3} = 0. \tag{84}$$

Thus, the stress state in the considered system will be investigated by the use of the equations (79)-(84). In this case it is assumed that the following boundary and contact conditions are satisfied.

$$Q'^{(1)}_{33}\Big|_{y'_3=0} = -P_0\delta(r')e^{i\omega t}\frac{1}{\left(\lambda^{(1)}\right)^2},\ Q'^{(1)}_{3r'}\Big|_{y'_3=0} = 0,$$

$$Q'^{(1)}_{33}\Big|_{y'_3=-h/\left(\lambda^{(1)}\right)^2} = Q'^{(2)}_{33}\Big|_{y'_3=-h/\left(\lambda^{(1)}\right)^2},\ Q'^{(1)}_{3r'}\Big|_{y'_3=-h/\left(\lambda^{(1)}\right)^2} = Q'^{(2)}_{3r'}\Big|_{y'_3=-h/\left(\lambda^{(1)}\right)^2},$$

$$u'^{(1)}_{r'}\Big|_{y'_3=-h/\left(\lambda^{(1)}\right)^2} = u'^{(2)}_{r'}\Big|_{y'_3=-h/\left(\lambda^{(1)}\right)^2},\ u'^{(1)}_{3}\Big|_{y'_3=-h/\left(\lambda^{(1)}\right)^2} = u'^{(2)}_{3}\Big|_{y'_3=-h/\left(\lambda^{(1)}\right)^2},$$

$$\left\{\left|Q'^{(2)}_{33}\right|;\left|Q'^{(2)}_{3r'}\right|;\left|Q'^{(2)}_{r'r'}\right|;\left|Q'^{(2)}_{\theta'\theta'}\right|;\left|Q'^{(2)}_{r'3}\right|;\left|u'^{(2)}_{3}\right|;\left|u'^{(2)}_{r'}\right|\right\} < M = \text{const} \quad \text{for} \quad y'_3 \to -\infty . \tag{85}$$

With the above-stated we exhaust the formulation of the problem which were investigated in [45, 46]. It should be noted that in the case where $\lambda^{(k)} = 1$ ($k = 1,2$), this formulation transforms to the corresponding ones of the classical linear theory of the elasticity for an incompressible body. For the solution to the formulated problem, according to [7,9], the following representation for the displacement and unknown function $p'^{(k)}$ is used:

$$u'^{(k)}_{r'} = -\frac{\partial^2}{\partial r'\partial y'_3}X'^{(k)},\ u'^{(k)}_{3} = \Delta'_1 X'^{(k)},\ p'^{(k)} = \Big[\left(\chi'^{(k)}_{1111} - \chi'^{(k)}_{1133} - \chi'^{(k)}_{1313}\right)\Delta'_1 +$$

$$\chi'^{(\underline{k})}_{3113}\frac{\partial^2}{\partial y'^2_3} - \rho'^{(\underline{k})}\frac{\partial^2}{\partial t^2}\Big]\frac{\partial}{\partial y'_3}X'^{(\underline{k})},\ \Delta'_1 = \frac{\partial^2}{\partial r'^2} + \frac{1}{r'}\frac{\partial}{\partial r'} \tag{86}$$

The function $X'^{(k)}$ in (86) satisfies the following equation:

$$\left[\left(\Delta'_1 + \left(\xi'^{(\underline{k})}_2\right)^2\frac{\partial^2}{\partial y'^2_3}\right)\left(\Delta'_1 + \left(\xi'^{(\underline{k})}_3\right)^2\frac{\partial^2}{\partial y'^2_3}\right) - \frac{\rho'^{(\underline{k})}}{\chi'^{(\underline{k})}_{1331}}\left(\Delta'_1 + \frac{\partial^2}{\partial y'^2_3}\right)\frac{\partial^2}{\partial t^2}\right]X'^{(\underline{k})} = 0, \tag{87}$$

where the constants $\xi'^{(k)}_2$ and $\xi'^{(k)}_3$ are determined through the mechanical constants and the initial strain state. The explicit form of the expressions for determination of these constants are given in the reviewed papers.

Thus, applying the presentation $g(r', y'_3, t) = \overline{g}(r', y'_3)e^{i\omega t}$ and applying the Hankel integral transformation (78) to the equations (79) – (87) the analytical expressions of the Hankel transforms of the sought values are determined. To determine their origin the above-discussed numerical algorithm is employed. The numerical investigations in the paper [45] were made for the case where $C^{(1)}_{10}\rho^{(2)}/C^{(2)}_{10}\rho^{(1)} = e$ and $e = C^{(2)}_{10}/C^{(1)}_{10}$. Here $C^{(2)}_{10}$ ($C^{(1)}_{10}$) is a mechanical constant of the half-space (covering layer) material. This constant enters the expression of the Treloar's potential (81). Note that the case considered in [45] corresponds to $\rho^{(2)} = \rho^{(1)}e^2$ which find various applications in many branches of modern engineering fields one of which, for example, is the bio-engineering. Moreover, note that the numerical investigations in [45] were made for $e = C^{(2)}_{10}/C^{(1)}_{10} \geq 1$, $0 \leq \Omega \leq 4$, where $\Omega^2 = (\omega h)^2\rho'^{(2)}/\left(2C^{(2)}_{10}\right)$. According to these numerical results, in [45] the influence of the initial finite strains on the dependencies between $Q'_{33}h^2/P_0$ $\left(Q'_{33} = Q'^{(1)}_{33}(0, -h/\left(\lambda^{(1)}\right)^2)\right)$ and Ω is studied. In particular, it is established that the pre-stretching (pre-compressing) of the layer (half-space) causes to decrease (increase) of the "resonance" frequency Ω_* and of the "resonance values of $Q'_{33}h^2/P_0$. This result is also illustrated by the graphs given in Fig. 18.

In the paper [46] the investigations were made for the case where $C_{10}^{(1)}/\rho'^{(1)} = C_{10}^{(2)}/\rho'^{(2)}$, $C_{10}^{(1)}/C_{10}^{(2)} \geq 1$. Note that the results which are similar in the qualitative sense to those obtained in [45] were also attained in the paper [46].

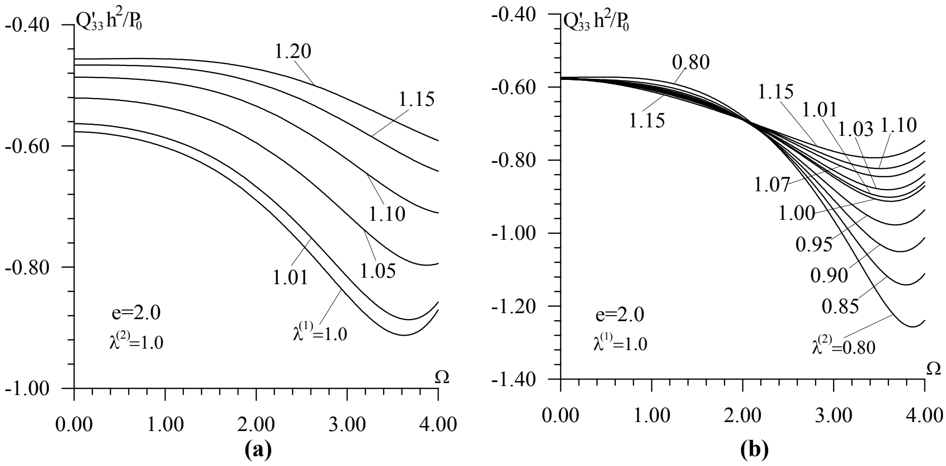


Figure 18. The influence of the pre-stretching of the covering layer (a) and the initial strains of the half-space (b) on the dependencies between $Q'_{33}h^2/P_0$ and Ω for the case where $e = C_{10}^{(2)}/C_{10}^{(1)} = 2.0$

In the foregoing investigations [23-26, 28, 35-38, 44-46] it was assumed that the region occupied by the body is semi-infinite. Therefore, the results obtained in these investigations cannot be applied, for example, in the cases where the afore-mentioned dynamical stress field is studied for the layered material, which rests on the rigid foundation. Nor these results can be applied for the structural elements whose basic material is covered with the layered ones. If the stiffness of the basic material (modulus of elasticity) is significantly greater than those of the covering layers, then the basic material can be modelled as a rigid foundation. It is well known that as a result of the covering procedure the residual (initial) stresses arise in the covering layers and it is almost inevitable to alert these stresses. Therefore, under studying the dynamical stress field in such structural members it is necessary to take the foregoing initial stresses into account.

Because of the above discussions in the papers [47, 48] the investigations carried out in the references [45, 46] were developed for systems, which comprise bilayered infinite slab and rigid foundation. It was assumed that the layers of the slab are finite pre-strained (stretched) radially. Moreover, it was assumed that the materials of the layers are incompressible neo-Hookean materials and the stress-strain relation for those are given through the Treloar potential. The investigations were made by applying the assumptions and notation used in [45, 46]. At the same time, the notation as h_1 and h_2 is also introduced, where h_1 (h_2) is a thickness of the upper (lower) layer of the slab, and the contact conditions in (85) were written at $y'_3 = -h_1/\left(\lambda^{(1)}\right)^2$, and the last condition in (85) was replaced by the following one.

$$u'^{(2)}_{r'}\Big|_{y'_3=-h_1/\left(\lambda^{(1)}\right)^2-h_2/\left(\lambda^{(2)}\right)^2}=0\,,\quad u'^{(2)}_{3}\Big|_{y'_3=-h_1/\left(\lambda^{(1)}\right)^2-h_2/\left(\lambda^{(2)}\right)^2}=0\,. \tag{88}$$

It was also introduced the notation

$$e=\frac{C_{10}^{(1)}}{C_{10}^{(2)}}\,,\ H=\frac{h_2}{h_1}\,,\ q_{33}^{(1)}=\left(\frac{Q'^{(1)}_{33}h_1^2}{P_0}\right)\Bigg|_{y'_3=-h_1/\left(\lambda^{(1)}\right)^2}\,,\ q_{33}^{(2)}=\left(\frac{Q'^{(2)}_{33}h_1^2}{P_0}\right)\Bigg|_{y'_3=-h_1/\left(\lambda^{(1)}\right)^2-h_2/\left(\lambda^{(2)}\right)^2} \tag{89}$$

In [47] it was assumed that $e\geq 1$, $C_{10}^{(1)}/\rho'^{(1)}=C_{10}^{(2)}/\rho'^{(2)}$ and the numerical investigations on the influence of the parameters H, e, $\lambda^{(1)}$ and $\lambda^{(2)}$ on the character of the dependencies among $q_{33}^{(1)}$, $q_{33}^{(2)}$ and Ω were made. Such numerical investigations for the case where $e\leq 1$, $\rho'^{(1)}=\rho'^{(2)}$ were made in the paper [48]. The similar investigations for the bilayered slab resting on a rigid foundation in the case where the elasticity relations of the layer's material are described by the Murnaghan potential was made in the paper [49]. Note that this investigation was made within the framework of the assumptions and notation used in the paper [44].

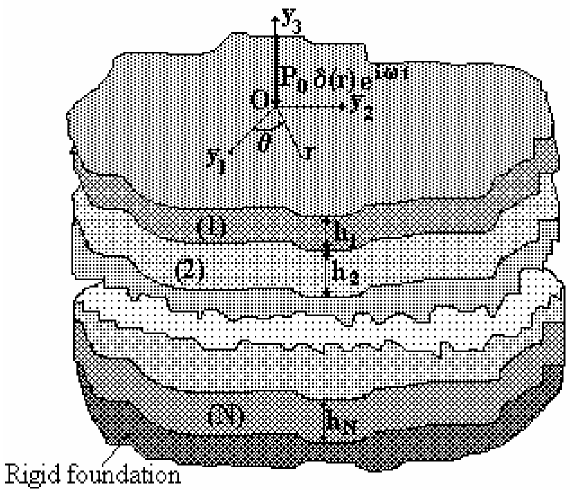


Figure 19. The structure of the many-layered slab resting on the rigid foundation

In the papers [50, 51] the investigations carried out in [47, 48] were developed for the many-layered slab. It is assumed that in the natural state the thicknesses of the layers are $h_1, h_2, \ldots, h_N$ (Fig.19). The problem formulation given in [47, 48] is generalized for the many-layered slab and in this case the equations (79)-(84) are satisfied for each layer shown in Fig. 19 and the conditions (85) are replaced by the following ones.

$$Q'^{(1)}_{33}\Big|_{y'_3=0} = -P_0\delta(r')e^{i\omega t}\frac{1}{\left(\lambda^{(1)}\right)^2}\,,\quad Q'^{(1)}_{3r'}\Big|_{y'_3=0} = 0\,,$$

$$\left\{Q'^{(1)}_{33}; Q'^{(1)}_{3r'}; u'^{(1)}_{3}; u'^{(1)}_{r'}\right\}\Big|_{y'_3=-h_1/\left(\lambda^{(1)}\right)^2} = \left\{Q'^{(2)}_{33}; Q'^{(2)}_{3r'}; u'^{(2)}_{3}; u'^{(2)}_{r'}\right\}\Big|_{y'_3=-h_1/\left(\lambda^{(1)}\right)^2}\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\left\{Q'^{(k-1)}_{33}; Q'^{(k-1)}_{3r'}; u'^{(k-1)}_{3}; u'^{(k-1)}_{r'}\right\}\Big|_{y'_3=-h_1/\left(\lambda^{(1)}\right)^2-h_2/\left(\lambda^{(2)}\right)^2-\ldots-h_{k-1}/\left(\lambda^{(k-1)}\right)^2} =$$

$$\left\{Q'^{(k)}_{33}; Q'^{(k)}_{3r'}; u'^{(k)}_{3}; u'^{(k)}_{r'}\right\}\Big|_{y'_3=-h_1/\left(\lambda^{(1)}\right)^2-h_2/\left(\lambda^{(2)}\right)^2-\ldots-h_{k-1}/\left(\lambda^{(k-1)}\right)^2}\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\left\{u'^{(N)}_{3}; u'^{(N)}_{r'}\right\}\Big|_{y'_3=-h_1/\left(\lambda^{(1)}\right)^2-h_2/\left(\lambda^{(2)}\right)^2-\ldots-h_{N-1}/\left(\lambda^{(N-1)}\right)^2-h_N/\left(\lambda^{(N)}\right)^2} = 0\,, \tag{90}$$

For numerical investigations the cases where $N = 2$, 4 and 6 are considered and it is assumed that the slab consists of the alternating layers of two materials, i.e., for example, for $N = 6$: $C^{(1)}_{10} = C^{(3)}_{10} = C^{(5)}_{10}$; $C^{(2)}_{10} = C^{(4)}_{10} = C^{(6)}_{10}$, $\rho'^{(1)} = \rho'^{(3)} = \rho'^{(5)}$, $\rho'^{(2)} = \rho'^{(4)} = \rho'^{(6)}$, $h_1 = h_3 = h_5$, $h_2 = h_4 = h_6$, $\lambda^{(1)} = \lambda^{(3)} = \lambda^{(5)}$, $\lambda^{(2)} = \lambda^{(4)} = \lambda^{(6)}$. In [50] the notation $e = C^{(2)}_{10}/C^{(1)}_{10}$, $H = h_2/h_1$ is also introduced and the case where $e \geq 1$ is considered. But in the paper [51] these investigations were made for the case $e \leq 1$. Note that in [50] the concrete numerical results were obtained under $e = 3.0$ for the dependencies among the $q^{(1)}_{33}$, $q^{(2)}_{33}$ (89) and Ω for the various N, $\lambda^{(1)}$, $\lambda^{(2)}$ and H. As an example, in Fig. 20 the graphs of these dependencies are given for the case where $\lambda^{(1)} = \lambda^{(2)} = 1.0$ (i.e. for the case where the initial pre-stretching in the layers is absent). The influence of the pre-stretching on these dependencies under $N = 6$ is given in Fig.21. The similar results for the case where $e = 1/3$ were obtained in the paper [51]. According to the analyses of the numerical results, the common character of the considered dependencies can be described as follows:

- the "resonance" frequency and the "resonance" values of the stresses decrease with increasing the number of the layers in the slab;
- for $\Omega \in (0, \Omega_*]$ the absolute values of $q^{(1)}_{33}$, $q^{(2)}_{33}$ increase monotonically with Ω and have the first local maximum under $\Omega = \Omega_*$;
- for $\Omega \in (\Omega_*, \Omega_j]$ the absolute values of $q^{(1)}_{33}, q^{(2)}_{33}$ decrease with Ω and have the first jump (discontinuity) under $\Omega = \Omega_j$;

for $\Omega > \Omega_j$ the values of $q^{(1)}_{33}, q^{(2)}_{33}$ have the following jumps (discontinuities) for certain values of Ω.

It should be noted that in [50, 51] it was assumed that the thickness of the slab increases with the increasing number of layers contained by the slab. In a practical as well as theoretical sense the case where the number of layers in the slab increases under constant thickness of the slab, i.e. the case where the number of layers in the slab increases by decreasing the thickness of the layers, has also a great significance. This situation is shown schematically in Fig.22. In connection with this, in the paper [52] the investigations [50, 51] were developed for the case shown in Fig. 22. It is assumed that in the natural state the thicknesses of the layers are $h_1, h_2, ..., h_N$ and $h_1 + h_2 + ... + h_N = H = \text{const.}$ Here H is a whole thickness of the slab which remains constant for any number of the layers (N) from which the slab is composed. The notation is introduced

$$e = \frac{C_{10}^{(1)}}{C_{10}^{(2)}}, \quad q_{33} = \left(\frac{Q'^{(N)}_{33} H^2}{P} \right)_{y'_3 = -\sum_{i=1}^{N} h_i / \left(\lambda^{(i)}\right)^2} . \qquad (91)$$

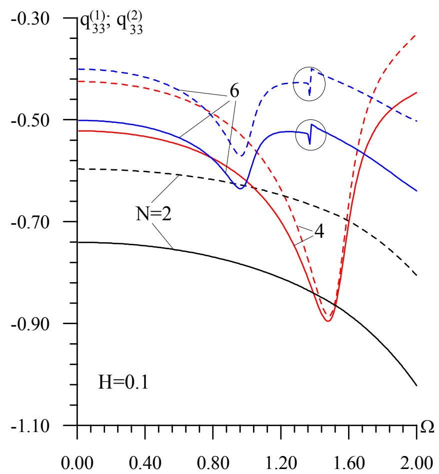


Figure 20. The influence of the number of the layers (N) on the dependencies among the $q_{33}^{(1)}$, $q_{33}^{(2)}$ and Ω

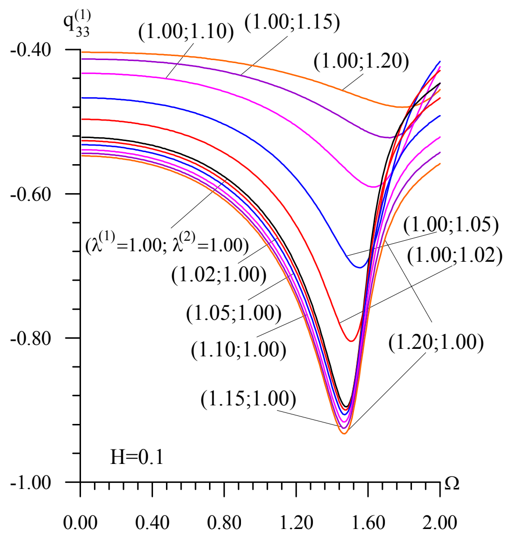


Figure 21. The influence of the pre-stretching of the layers on the dependencies among the $q_{33}^{(1)}$, $q_{33}^{(2)}$ and Ω under $N = 6$

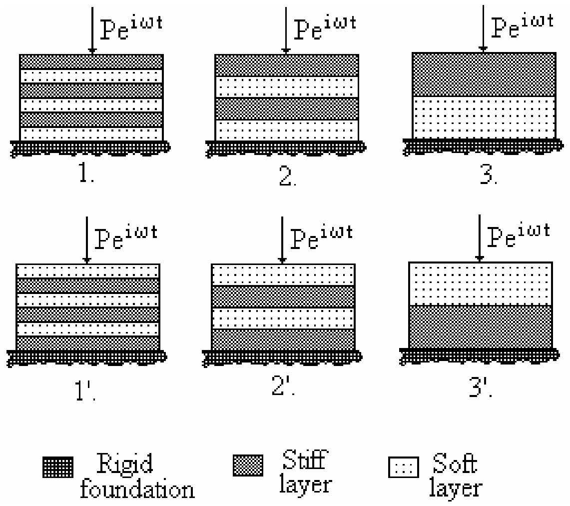


Figure 22. The structure of the many-layered slab with constant thickness

The numerical results were obtained for the case where $h_i = H/N$, $C_{10}^{(2)}\rho'^{(1)}/\left(C_{10}^{(1)}\rho'^{(2)}\right)=1$. As it follows from Fig.22 that the cases 1., 2. and 3. (1'., 2'. and 3'.) correspond to $e>1$ $(e<1)$. As in [50, 51] the numerical investigations were made for the dependencies between q_{33} and Ω under $N=2,4$ and 6 for various values of the problem parameters. As a result of this investigation, in particular, the following conclusions were established.

- the location sequence of the soft and stiff layers in the slab can change significantly the character of the frequency response of that;
- the absolute values of $q_{33}\big|_{\Omega=\Omega_*}$ (91) decrease but the values of Ω_* (for which $dq_{33}/d\Omega = 0$) increase with the pre-stretching of the stiff layers, but the pre-stretching of the soft layers causes the case where $dq_{33}/d\Omega = 0$ to disappear and to appear the resonance jumping in the values of q_{33} (91) under considered change range of Ω.

It is known that the dynamical behaviour of the elements of constructions is very sensitive to a violation of the completeness of the contact conditions. Namely, by the use of this sensitivity the ultrasonic nondestructive methods were developed for determination of the various types of defects in structural members. In connection with this, the theoretical investigation of the influence of the violation of the completeness of the contact conditions on the frequency response of the many-layered slab on a rigid foundation has a great significance in both theoretical and practical sense. According to these discussions, in the paper [53] the investigation carried out in [52] is developed for the case where the incomplete contact conditions are satisfied between the slab and rigid foundation. In other words, in [53] the last condition in (90) is replaced by the following one.

$$Q'^{(k)}_{3r}\Big|_{y'_3=-h_1/\left(\lambda^{(1)}\right)^2-h_2/\left(\lambda^{(2)}\right)^2-\ldots-h_{N-1}/\left(\lambda^{(N-1)}\right)^2-h_N/\left(\lambda^{(N)}\right)^2}=0$$

$$u'^{(k)}_{r'}\Big|_{y'_3=-h_1/\left(\lambda^{(1)}\right)^2-h_2/\left(\lambda^{(2)}\right)^2-\ldots-h_{N-1}/\left(\lambda^{(N-1)}\right)^2-h_N/\left(\lambda^{(N)}\right)^2}=0\,. \tag{92}$$

Numerical investigations were made for the case 1 and 1' (Fig. 22) under $C_{10}^{(2)}\rho'^{(1)} / \left(C_{10}^{(1)}\rho'^{(2)}\right) = 1$. These investigations show that the incompleteness of the contact conditions the absolute maximum values of q_{33}, Ω_* and Ω_j decrease. The influence of the incompleteness on the values q_{33} becomes more significant as $\Omega \to \Omega_j$.

It follows from the foregoing review that in the considered papers the investigations of the dynamical problems for the initially stressed bodies by the use of the TLTEWISB were made for the homogeneous or piecewise homogeneous medium. Moreover, in these investigations the concrete results are obtained for the cases where the afore-mentioned initial strains are homogeneous ones. Consequently, in these papers there is not any investigation of dynamical problems for initially stressed continuously inhomogeneous medium. At present such problems become more urgent in connection with arising and applications of a new class of materials called "Functionally Graded Materials" (FGM). These materials are multi-phase ones with volume fractions of the constituents varying gradually in a pre-determined profile. In recent years a large number of theoretical, computational and experimental studies have been carried out to understand the mechanical behaviour of structural members from FGM (see references [54-58] for review and further references). In these investigations FGM is modelled as continuously inhomogeneous materials. As usual, it is assumed that the elastic modulus of these materials is a function of the coordinate directed along the thickness and this function is taken as a polynomial. According to the theoretical results obtained for various values of the coefficients and power of this polynomial FGM with required properties can be determined. Among these results the dynamical ones have a special importance because two of the main application fields of the FGM is the aerospace and mechanical engineering.

It should be noted that in many cases FGM are used as a shielding (covering) layer for the basic materials of the structural members. As it has been noted above, as far as the manufacturing procedure of this layer is concerned the initial (or residual) strains and stresses arise in that. Therefore the theoretical investigations of the dynamical behaviour of the afore-mentioned covering FGM layer with initial stresses has a high significance in both fundamental and applied sense. Taking the above discussions into account in the paper [59] the attempt was made to study the frequency response (amplitude-frequency relation) of the axisymmetrically finite pre-stretched slab from incompressible FGM on a rigid foundation by the use of the TLTEWISB. The elasticity relations of the slab material were given through the Treloar potential. Moreover, it was assumed that on the free face plane of the slab the time-harmonic point-located force acts and the mechanical properties of the slab material properties are continuous functions of a coordinate directed along the thickness of that only. According to this situation, the equations of the TLTEWISB become the equations with variable coefficients even under homogeneous initial strain state and therefore to find an analytical solution to these equations for the corresponding boundary and contact conditions becomes more complicated. In the paper [59] the discrete-analytical method was proposed for the solution to the considered boundary value problems and the like. According to this method the continuously inhomogeneous material is replaced by the piecewise homogeneous (layered) material and the mechanical properties of each layer is determined through its geometrical location along the thickness and by the function describing the change of the mechanical properties of the slab material. The analytical solution to the equations written for each layer is determined by employing the method discussed above. In this case a number of the layers by which the slab is modelled, are determined from the convergence requirement of the numerical results. Numerical results on the influence of the function describing inhomogeneous mechanical properties of the slab material and the pre-stretching of the slab on the frequency response of that are presented and analyzed.

It is evident that the initial stresses in the members of constructions in many cases, specially, in the cases where these members contain a source for stress concentration, are inhomogeneous. In such cases the investigations of the time-harmonic stress-state problems within the framework of the TLTEWISB has also a great significance in the theoretical and practical sense. However, in the mentioned cases as a result of the inhomogeneous distribution of the initial (or residual) stress state the corresponding boundary value problems cannot be solved by employing of the above-described analytical method. Consequently, for the solution to such problems it is necessary to develop the corresponding numerical method. In connection with this in the paper [60] the FEM was developed and applied for the investigation of the stress distribution in the pre-stretched simply supported strip containing two neighbouring circular holes under forced vibration. The numerical results on the influence of the initial stretched static forces on the stress concentration around the holes under forced vibration are presented. Note that the related static problem was investigated in the paper [61].

Thus, with the above-stated we exhaust the consideration of the investigations regarding the dynamical stress field in the initially stressed bodies.

4. CONCLUSIONS: AREAS OF FURTHER RESEARCH

In the present paper the review of the recent investigations regarding the dynamical problems of the bodies with initial stresses was considered. In this case the investigations carried out in the recent six years within the framework of the piecewise homogeneous bodies model was considered with the use of the TLTEISB and the main attention was focused on the studies made by the author and his students. The researches on the wave propagation and on the dynamical time-harmonic stress state problems were reviewed separately.

In the opinion of the author, in view of the increased requirements of applied and theoretical problems, future investigations on the dynamical problems for the initial stressed bodies in following areas are necessary.

1. The wave propagation (dispersion) in the unidirected fibrous composites with initial stresses for the high concentrations of the fibers therein.
2. The wave propagation (dispersion) in the packet of layers with initial stresses under various boundary conditions satisfied on the face planes of the packet.
3. Nonstationary Lamb's problems for the layered half-space with initial stresses.
4. Nonstationary dynamical problems on the stress-state in the packet of the layers under various boundary conditions satisfied on the face planes of the packet.

At the same time, it is also necessary to develop the investigations regarding the time-harmonic stress-state in the elements of constructions with initial (residual) stresses.

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