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A NEW 2+1-DIMENSIONAL HAMILTONIAN INTEGRABLE SYSTEM

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ABSTRACT

It is shown that a new 2+1-dimensional second-order partial differential equation, when written as a first-order nonlinear evolutionary system, admits bi-Hamiltonian structure. Therefore, by Magri's theorem it is a completely integrable system. For this system a Lagrangian is introduced and Dirac's theory is applied in order to obtain first Hamiltonian structure. Then recursion operator is constructed and finally the second Hamiltonian structure for this system is obtained. Jacobi identity for the Hamiltonian structure is proved by using Olver's method. Thus, it is an example of a completely integrable system in three dimensions.
Keywords: Integrable systems, Hamiltonian integrable systems.
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2+1 BOYUTTA YENİ İNTEGRE EDİLEBİLİR HAMİLTONIAN SİSTEMLER

ÖZET

Yeni 2+1 boyutlu ikinci mertebeden kismi türevli diferansiyel denklem, birinci mertebe lineer olmayan değişim sistemi olarak yazıldığında, bu yeni sistemin bi-Hamiltonian yapıya sahip olduğu gösterilmiştir. Böylece Magri teoremine göre tamamen integere edilebilir bir sistem elde edilmiştir. Bu sistem için Lagrangian elde edilmiş, ve birinci Hamiltonian yapıyı elde etmek için Dirac teori uygulanmıştır. Sistem için tekrarlama (recursion) operatorü kurulmuş ve son olarak ikinci Hamiltonian yapı elde edilmiştir. Hamiltonian yapılar için Jacobi özdeşliği Olver'in metodu kullanılarak ispatlanmıştır. Böylece yeni denklem üç boyutta tamamen integre edilebilir sitemlere bir örnek teşkil etmektedir.
Anahtar Sözcükler: İntegre edilebilir sistemler, Hamiltonian integer edilebilir sistemler.

1. INTRODUCTION

Integrable Hamiltonian systems are studied for more than three decades and there are many examples of 1+1-dimensional ones in literature. The well-known example in this field is Korteweg- de Vries (KdV) equation. The first discovery, made by Gardner [1], was that the KdV equation could be written as a completely integrable Hamiltonian system. This idea was further developed by Zakharov and Fadeev [2]. The general concept of a Hamiltonian system of evolution equations first appears in the works of Magri [3], Kupershmidt [4] and Manin [5]. Further developments, including the simplified techniques for verifying the Jacobi identity, appear in Gelfand and Dorfman [6], Olver [7] and Kosmann –Schwarzbach [8]. The basic

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theorem on bi-Hamiltonian systems is due to Magri [3, 9], who was also the first one to publish the second Hamiltonian structure for the KdV and the other equations.

For a long time there were only few examples of 2+1- and even no examples of 3+1-dimensional integrable systems. Very recently Neyzi, Nutku and Sheftel [10] discovered that the second heavenly equation of Plebanski, when being presented in a two component form, is a 3+1-dimensional bi-Hamiltonian integrable system. Later, it was discovered that the complex Monge-Ampere equation in (3+1) real dimensions is completely integrable in the sense of the Magri's theorem [11].

In [12] we studied symmetry reduction of second heavenly equation and we obtained a 2+1-dimensional bi-Hamiltonian system. In this paper I will present a new 2+1-dimensional bi-Hamiltonian system. The Lagrangian of this system is,

$$L = u_t^2 u_{xx} - u_x^2 - u_x u_y - \alpha u_t \left(u_x - u_y\right) \tag{1}$$

and Euler Lagrange equation gives the following non linear 2+1 dimensional partial differential equation in one-component form.

$$u_{tt} u_{xx} - u_{tx}^2 - u_{xx} - u_{xy} - \alpha\left(u_{xt} - u_{yt}\right) = 0 \tag{2}$$

where α is an arbitrary constant. This system is obtained by using a linear combination of invariants given in [12].

In section 2, I will give Lagrangians and construct the first Hamiltonian structure using Dirac's theory [13] of constraints. In section 3 I derive a recursion operator for a new system. In section 4 I obtain the second Hamiltonian structure and Hamiltonian function by applying the recursion operator to the first Hamiltonian structure. Finally in section 5, the Jacobi identity for the Hamiltonian structure will be checked in detail by using Olver's method [14].

2. LAGRANGIAN AND FIRST HAMILTONIAN STRUCTURE

In this part I use the method of [10] for the calculation of the first Hamiltonian structure. The Lagrangian density (1) for the equation in one-component form (2), but this must be converted to a form suitable for applying Dirac's theory of constraints. For this purpose, I introduce an auxiliary variable q whereby system (2) assumes the form

$$\begin{cases} u_t = q \\ q_t = \dfrac{1}{u_{xx}}\left[q_x^2 + \alpha\left(q_x - q_y\right) + u_{xx} - u_{xy}\right] \equiv Q \end{cases} \tag{3}$$

of a first-order two-component system. Here sub indexes x, y and t stand for partial derivative of $\dfrac{\partial}{\partial x}$, $\dfrac{\partial}{\partial y}$ and $\dfrac{\partial}{\partial t}$ respectively and in all paper I will use the same notation. Lagrangian density for system (3) is given by, should be degenerate, that is, linear in the time derivative of the unknown u_t and with no q_t :

$$L = \frac{1}{2}\left(2 q u_t u_{xx} - q^2 u_{xx} - u_x^2 - u_x u_y - \alpha u_t\left(u_x - u_y\right)\right) \tag{4}$$

This Lagrangian is degenerate [16], because its Hessian

$$\frac{\partial^2 L}{\partial u_t^2} = 0$$

vanishes identically. Alternatively, the canonical momentum given by;

$$\Pi = \frac{\partial L}{\partial u_t} = qu_{xx} - \alpha\left(u_x - u_y\right)$$

can not be inverted the velocity u_t and we have degenerate Lagrangian. After substituting $q = u_t$, coincides with our original Lagrangian (1) up to a total divergence. We can easily check that Euler-Lagrange equations for (4) give the system (3). The variational derivative for u^k defined as following.

$$\frac{\delta}{\delta u^k} = \frac{\partial}{\partial u^k} - \partial_t \frac{\partial}{\partial u_t^k} - \partial_x \frac{\partial}{\partial u_x^k} - \partial_y \frac{\partial}{\partial u_y^k} + \partial_x^2 \frac{\partial}{\partial u_{xx}^k} + \partial_y^2 \frac{\partial}{\partial u_{yy}^k} - \cdots$$

Here $k = 1,2$ with $u^1 = q$ and $u^2 = u$, hence we get,

$$\frac{\delta L}{\delta q} = u_t u_{xx} - qu_{xx} = 0 \Rightarrow u_t = q$$

and

$$\frac{\delta L}{\delta u} = -q_t u_{xx} + \alpha\left(q_x - q_y\right) + u_{xx} + u_{xy} + q_x^2 = 0$$

$$q_t = \frac{1}{u_{xx}}\left[q_x^2 + \alpha\left(q_x - q_y\right) + u_{xx} - u_{xy}\right]$$

Since the Lagrangian density (4) is linear in u_t and has no q_t, the canonical momenta

$$\begin{aligned} \pi_u &= \frac{\partial L}{\partial u_t} = 2qu_{xx} - \alpha\left(u_x - u_y\right) \\ \pi_q &= \frac{\partial L}{\partial q_t} = 0 \end{aligned} \tag{5}$$

cannot be inverted for the velocities u_t and q_t and so the Lagrangian is degenerate. Therefore, according to the Dirac's theory [13], we impose (5) as constraints

$$\begin{aligned} \phi_u &= \pi_u - 2qu_{xx} + \alpha\left(u_x - u_y\right) \\ \phi_q &= \pi_q \end{aligned} \tag{6}$$

where the canonical momenta should satisfy canonical Possion brackets

$$\left[\pi_i(x,y), u^k(x',y')\right] = \delta_i^k \delta(x-x')\delta(y-y')\ ,\ i,k=1,2$$

and calculate the Passion brackets of the constraints

$$K_{ij} = \left[\phi_i(x,y), \phi_j(x',y')\right]. \tag{7}$$

If we organize them into a 2×2 matrix form, we find

$$\begin{aligned} K_{11} &= \left[\phi_u(x,y), \phi(x',y')\right] = -2q(x')\delta_{x'x'}(x-x')\delta(y-y') \\ &+ 2q(x)\delta_{xx}(x'-x)\delta(y'-y) + \alpha\delta_{x'}(x-x')\delta(y-y') \\ &- \alpha\delta(x-x')\delta_{y'}(y-y') + \alpha\delta_x(x'-x)\delta(y'-y) - \alpha\delta(x'-x)\delta_y(y'-y) \end{aligned} \tag{8}$$

$$K_{12} = -K_{21} = \left[\phi_u(x,y), \phi_q(x',y')\right] = -2u_{xx}\delta(x'-x)\delta(y'-y)$$

$$K_{22} = \left[\phi_q(x,y), \phi_q(x',y')\right] = 0\ ,$$

where the subscripts run from 1 to 2 with 1 and 2 corresponding to u and q, respectively. In all the coefficients of K_{ij} if we kill factor (-2) this yields the symplectic operator K_{ij} that is an inverse of the Hamiltonian operator J_0 :

$$K_{ij} = \begin{pmatrix} D_x q + q_x D_x - \alpha\left(D_x - D_y\right) & -u_{xx} \\ u_{xx} & 0 \end{pmatrix}, \tag{9}$$

which is an explicitly skew-symmetric local matrix-differential operator. The first Hamiltonian operator $J_0 = \left(K_{ij}\right)^{-1}$ is obtained by inverting K_{ij} in (9) as

$$J_0 = \begin{pmatrix} 0 & \dfrac{1}{u_{xx}} \\ -\dfrac{1}{u_{xx}} & \dfrac{q_x}{u_{xx}^2}D_x + D_x\dfrac{q_x}{u_{xx}^2} - \dfrac{\alpha}{u_{xx}}\left(D_x - D_y\right)\dfrac{\alpha}{u_{xx}} \end{pmatrix} \tag{10}$$

which is explicitly skew-symmetric. Also it satisfies the Jacobi identity, as will be shown in detail in section 5. The Hamiltonian density is

$$H_1 = \pi_u u_t + \pi_q q_t - L\ ,$$

which results in

$$H_1 = \frac{1}{2}\left(q^2 u_{xx} + u_x^2 - u_x u_y\right). \tag{11}$$

The new system (3) can now be written in a Hamiltonian form with the Hamiltonian density H_1 defined by (11)

$$\begin{pmatrix} u_t \\ q_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u \mathrm{H}_1 \\ \delta_q \mathrm{H}_1 \end{pmatrix} = \begin{pmatrix} q \\ \frac{1}{u_{xx}}\left[q_x^2 + \alpha\left(q_x - q_y\right) + u_{xx} - u_{xy}\right] \end{pmatrix} \tag{12}$$

where $\delta_u = \frac{\delta}{\delta u}$ and $\delta_q = \frac{\delta}{\delta q}$ are Euler-Lagrange operators [14], defined as

$$E_u\left(H_1\right) = \frac{\delta \mathrm{H}_1}{\delta u} = \sum_{j=0}^{\infty}\left(-\frac{d}{dx}\right)^j \frac{\partial \mathrm{H}_1}{\partial u_j} \tag{13}$$

with $u_j = \frac{du}{dx^j}$, and similarly for $\delta_q \mathrm{H}_1$, which correspond to the variational derivatives of the Hamiltonian functional $\mathrm{H}_1 = \int_{-\infty}^{\infty} H_1 dxdy$.

3. RECURSION OPERATOR

We start with the equation determining symmetries of the two-component system (3). We introduce the two-component symmetry characteristic Φ by

$$\begin{array}{l} u_\tau = \varphi\left(x, y, t, u, q, u_x, u_y, q_x, q_y\right) \\ q_\tau = \psi\left(x, y, t, u, q, u_x, u_y, q_x, q_y\right) \end{array}, \qquad \Phi \equiv \begin{pmatrix} \varphi \\ \psi \end{pmatrix}. \tag{14}$$

From the Frechét derivative of the flow we find

$$A = \begin{pmatrix} D_t & -1 \\ \frac{Q-1}{u_{xx}} D_x^2 + \frac{1}{u_{xx}} D_x D_y & D_t - \frac{2q_x}{u_{xx}} D_x - \frac{\alpha}{u_{xx}}\left(D_x - D_y\right) \end{pmatrix} \tag{15}$$

So that the equation determining symmetries of the new three-dimensional evolution system is given by

$$A(\Phi) = 0. \tag{16}$$

If we combine the first determining equation with the second equation in (16), multiplied by the overall factor u_{xx}, we reproduce the determining equation for symmetries of original equation (1). The equation for symmetries (16) can be set in a 2-term divergence form

$$\left(q_x \varphi_x - \alpha\left(\varphi_x - \varphi_y\right) + u_{xx}\psi\right)_t - \left(\left(1 - q_t\right)\varphi_x + q_x \psi - \varphi_y\right)_x = 0 \tag{17}$$

that implies the local existence of the potential variable $\tilde{\varphi}$ defined by

$$\begin{aligned} \tilde{\varphi}_x &= q_x\varphi_x - \alpha(\varphi_x - \varphi_y) + u_{xx}\psi \\ \tilde{\varphi}_t &= (1-q_t)\varphi_x + q_x\psi - \varphi_y \end{aligned} \tag{18}$$

which also satisfies the same determining equation for the symmetries of (1) and therefore it is a partner symmetry for φ. In the two-component form, we define the second component of this new symmetry, similar to the definition of ψ, as $\tilde{\psi} = \tilde{\varphi}_t$. Then the two-component

$$\tilde{\Phi} = \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix}$$

vector satisfies the determining equation for symmetries in the form (16) and hence a symmetry characteristic of the system (3), provided the vector (14) is also a symmetry characteristic.

$$\tilde{\Phi} = R(\Phi) \tag{19}$$

with the recursion operator R given by

$$R = \begin{pmatrix} D_x^{-1}\left[q_x D_x + \alpha(D_x - D_y)\right] & D_x^{-1}u_{xx} \\ (Q-1)D_x + D_y & -q_{xx} \end{pmatrix}, \tag{20}$$

where D_x^{-1} is the inverse of D_x and defined as

$$D_x^{-1} f = \frac{1}{2}\left(\int_{-\infty}^{x} - \int_{x}^{\infty}\right) f(\xi)d\xi. \tag{21}$$

For the properties of this operator see [15]. Moreover, vanishing of the commutator $[R, A]$, computed without using the equations (3), reproduce the new system (3) and hence the operator R and A form a Lax pair for the 2-component system. The commutator reads,

$$[R, A] = \begin{pmatrix} D_x^{-1}(q_t - Q)_{xx} - (q_t - Q)_x & D_x^{-1}(u_t - q)_{xx} \\ \dfrac{1}{u_{xx}}\left\{(Q-1)(u_t - q)_{xx} + \left(\alpha(D_x - D_y) - 2q_x\right)(q_t - Q)\right\}D_x & (q_t - Q)_x \end{pmatrix}$$

It can be easily see that $[R, A] = 0$ is equivalent to the system (3) and therefore R and A form a Lax pair for 2-component system (3).

4. SECOND HAMILTONIAN STRUCTURE AND HAMILTONIAN FUNCTION

The second Hamiltonian operator J_1 is obtained by applying the recursion operator (20) to the first Hamiltonian operator $J_1 = RJ_0$ with the result

$$J_1 = \begin{pmatrix} D_x^{-1} & -\dfrac{q_x}{u_{xx}} \\ \dfrac{q_x}{u_{xx}} & J_1^{22} \end{pmatrix}, \tag{22}$$

where

$$J_1^{22} = \frac{1}{2}\left((Q-1) D_x \frac{1}{u_{xx}} + \frac{1}{u_{xx}} D_x (Q-1) \right) + \frac{1}{2}\left(D_y \frac{1}{u_{xx}} + \frac{1}{u_{xx}} D_y \right) - \frac{2q_x}{u_{xx}} D_x \frac{q_x}{u_{xx}}$$
$$+ \frac{\alpha}{2}\left(\frac{q_x}{u_{xx}} (D_y - D_x) \frac{1}{u_{xx}} + \frac{1}{u_{xx}} (D_y - D_x) \frac{q_x}{u_{xx}} \right).$$

Operator J_1 is obviously skew-symmetric and the Jacobi identity for this operator will be checked in the next section in detail. The Hamiltonian function for J_1 which generates the system (3) is given by

$$H_0 = (x+y) q u_{xx}. \tag{23}$$

The Hamiltonian function H_0 satisfies the recursion relation of Magri,

$$\begin{pmatrix} u \\ q \end{pmatrix}_t = J_0 \begin{pmatrix} \delta_u \mathrm{H}_1 \\ \delta_q \mathrm{H}_1 \end{pmatrix} = J_1 \begin{pmatrix} \delta_u \mathrm{H}_0 \\ \delta_q \mathrm{H}_0 \end{pmatrix} \tag{24}$$

which shows that the new equation (1) in the 2-component form (3) is a bi-Hamiltonian system. The second Hamiltonian operator is obtained by acting with the recursion operator R on the Hamiltonian operator J_0. In order to have the higher flows we generalize this relation as

$$J_n = R^n J_0 \tag{25}$$

In the case of (22) we have $n = 1$. If we take, for example, $n = 2$ we can generate a new Hamiltonian operator $J_2 = R^2 J_0 = RJ_1 = J_1 J_0^{-1} J_1$. Here we used the relation $R = J_1 J_0^{-1}$. By the repeated application of the recursion operator (20) to the Hamiltonian operators J_0, J_1 and so on, we could obtain multi-Hamiltonian representation of our new system.

5. JACOBI IDENTITY

In this section, I will concentrate on checking of the Jacobi identity for the Hamiltonian operators J_0 and J_1.

Definition: A linear operator $J : A^q \to A^q$ is called *Hamiltonian* if its Possion bracket $\{P,Q\} = \int \delta P \cdot J\delta Q\, dx$ satisfies *skew-symmetry* property

$$\{P,Q\} = -\{Q,P\}, \tag{26}$$

and the *Jacobi identity*

$$\{\{P,Q\},R\} + \{\{R,P\},Q\} + \{\{Q,R\},P\} = 0 \tag{27}$$

for all functionals P, Q and R

But using this definition directly, the verification of Jacobi identity (27), even for simplest skew-adjoint operators, appears a hopelessly complicated computational task. For this reason we will use the Olver's method [14] by following the theorem below.

Theorem: *Let J be a skew-adjoint $q \times q$ matrix differential operator and $\Theta = \frac{1}{2}\int (J\theta \wedge \theta)dx$ be the corresponding bi-vector. Then J is Hamiltonian if and only if*

$$\mathrm{Pr}\, V_{J\theta}(\Theta) = 0. \tag{28}$$

We mentioned that if we can present the system (3) in the form (24), the system is called *bi-Hamiltonian* system. We say that J_0, J_1 form a Hamiltonian pair if every linear combination $aJ_0 + bJ_1$ where a and b are constants, should satisfy the Jacobi identity. Therefore, if we directly compute the Jacobi identity for $\Gamma = aJ_0 + bJ_1$, then we guarantee that J_0 and J_1 satisfy the Jacobi identity. Because, if we choose $a = 1, b = 0$ and $a = 0, b = 1$ then we will end up with the Jacobi identity for J_0 and J_1 respectively. In this way, we will prove that, J_0 and J_1 independently satisfy the Jacobi identity and also that any linear combination $aJ_0 + bJ_1$ also satisfies Jacobi identity. Therefore we start with,

$$\Gamma = aJ_0 + bJ_1 = \begin{pmatrix} bD_x^{-1} & -\dfrac{bq_x - a}{u_{xx}} \\ \dfrac{bq_x - a}{u_{xx}} & -AD_x + BD_y + C \end{pmatrix} \tag{29}$$

where

$$A=\frac{1}{u_{xx}^{2}}\left(bq_x^2-2aq_x+\alpha bq_y+bu_{xy}-\alpha a\right)$$

$$B=\left(\frac{b}{u_{xx}}+\frac{\alpha\left(bq_x-a\right)}{u_{xx}^{2}}\right) \tag{30}$$

$$C=\frac{u_{xxx}}{u_{xx}}A-\frac{u_{xxy}}{u_{xx}}B+\left(a-bq_x\right)\frac{q_{xx}}{u_{xx}^{2}}.$$

By using theorem (28), we define the two-form bi-vector

$$\Theta=\frac{1}{2}\int\sum_{i,j}\left(\Gamma_{ij}\omega^{i}\right)\wedge\omega^{j}\,dx\,dy \tag{31}$$

Where the uni-vectors correspond to $\omega^1=\eta$ and $\omega^2=\theta$ and $i,j=1,2$. Hence (31) becomes

$$\Theta=\frac{1}{2}\int\left(\left(D_x^{-1}\eta\right)\wedge\eta+\left(\frac{\alpha}{u_{xx}}-\frac{bq_x}{u_{xx}}\right)\theta\wedge\eta+\left(\frac{bq_x}{u_{xx}}-\frac{a}{u_{xx}}\right)\eta\wedge\theta-A\theta_x\wedge\theta+B\theta_y\wedge\theta\right)dxdy$$

If we substitute Θ in (28) we obtain

$$\begin{aligned}\Pr V_{\Gamma\omega}(\Theta)=\frac{1}{2}\int\Bigg\{&\left(-\frac{2b}{u_{xx}}\Pr V_{J\omega}(q_x)-2(bq_x-a)\Pr V_{J\omega}\left(\frac{1}{u_{xx}}\right)\right)\wedge\theta\wedge\eta\\&-\frac{1}{u_{xx}^{2}}\left(2(bq_x-a)\Pr V_{J\omega}(q_x)+\alpha b\Pr V_{J\omega}(q_y)+b\Pr V_{J\omega}(u_{xy})-u_{xx}^{2}A\Pr V_{J\omega}\left(\frac{1}{u_{xx}^{2}}\right)\right)\wedge\theta_x\wedge\theta\\&\left(b\Pr V_{J\omega}\left(\frac{1}{u_{xx}}\right)+\frac{b\alpha}{u_{xx}^{2}}\Pr V_{J\omega}(q_x)+\alpha(bq_x-a)\Pr V_{J\omega}\left(\frac{1}{u_{xx}^{2}}\right)\right)\wedge\theta_y\wedge\theta\Bigg\}dx\,dy\end{aligned} \tag{32}$$

and by using the following relation [14],

$$\Pr V_{\Gamma\omega}=\sum_{\alpha,\beta,J}D_J\left(\sum\Gamma_{\alpha\beta}\omega^{\beta}\right)\frac{\partial}{\partial u_J^{\alpha}}\qquad J=1,x,xx,xxx,\cdots \tag{33}$$

we can compute the terms given in (32) as given below

$$\Pr V_{J\omega}(q_x)=D_x\left(\left(\frac{bq_x-a}{u_{xx}}\right)\eta-A\theta_x+B\theta_y+C\theta\right)$$

$$\Pr V_{J\omega}(q_y)=D_y\left(\left(\frac{bq_x-a}{u_{xx}}\right)\eta-A\theta_x+B\theta_y+C\theta\right)$$

$$\Pr V_{J\omega}\left(\frac{1}{u_{xx}}\right)=-\frac{1}{u_{xx}^{2}}D_x\left(b\eta-\left(\frac{bq_x-a}{u_{xx}}\right)_x\theta-\left(\frac{bq_x-a}{u_{xx}}\right)\theta_x\right)$$

$$\Pr V_{J\omega}\left(\frac{1}{u_{xx}^{2}}\right)=-\frac{2}{u_{xx}^{3}}D_x\left(b\eta-\left(\frac{bq_x-a}{u_{xx}}\right)_x\theta-\left(\frac{bq_x-a}{u_{xx}}\right)\theta_x\right)$$

$$\Pr V_{J\omega}\left(u_{xy}\right)=D_y\left(b\eta-\left(\frac{bq_x-a}{u_{xx}}\right)_x\theta-\left(\frac{bq_x-a}{u_{xx}}\right)\theta_x\right).$$

After we substitute these terms in (32) we get a very large equation which is not suitable to write here. After that we do a very lengthy and cumbersome calculation and finally we get zero. This means, by virtue of (28), that the Jacobi identity is satisfied both for J_0 and J_1 and they form a Hamiltonian pair.

6. CONCLUSION

We discover new 2+1-dimensional nonlinear evolution equation and we write this equation in a two-component form in order to obtain its Hamiltonian structure. We start with the first Hamiltonian structure. We use Dirac's theory of constraints to construct the matrix operator K which is an inverse of the first Hamiltonian operator J_0. We obtain a recursion operator for symmetries and we show that the recursion operator R and the linear operator A of the equation determining symmetries commute and, moreover, they form a Lax pair for the new two-component evolutionary system. We have found second Hamiltonian structure by acting with the recursion operator R on the first Hamiltonian operator J_0. Finally, we prove that both Hamiltonian operators J_0, J_1 and also their linear combination aJ_1+bJ_2 satisfy the Jacobi identity. Therefore, this new system is bi-Hamiltonian and, by Magri's theorem, the multi-Hamiltonian structure makes the new system to be a completely integrable system in 2+1-dimensions. In the future work we will present the Lie algebra of all point symmetries and integrals of motion which generate all variational point symmetries of the new evolution system.

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