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A DECOMPOSITION OF CONTINUITY ON F*– SPACES AND MAPPINGS ON SA*– SPACES

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Abstract: An ideal topological space (X,τ,I) is said to be an F* – space if A=Cl*(A) for every open set A ⊂ X. In this paper, a decomposition of continuity on F* – spaces is introduced. An ideal topological space (X,τ,I) is said to be an SA* – space if (A)*⊂ A for every set A⊂X. It is shown that δ_I – r – continuity (resp. pre – I – continuity, semi – δ – I – continuity, * – perfect continuity) is equivalent to R – I – continuity (resp. R – I – continuity, t – I – continuity, * – dense – in – itself continuity) if the domain is an SA* – space.

Key words: R – I – open set, δ – I – open set, δ – I – regüler set, decomposition of R – I – continuity, topological ideal.

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F*-UZAYLARDA SÜREKLİLİĞİN BİR AYRIŞIMI VE SA* -UZAYLARDA DÖNÜŞÜMLER

Özet: Eğer (X, τ, I) uzayının her açık A alt kümesi için A = Cl*(A) ise bu taktirde bu uzaya F* – uzay denir. Bu çalışmada, F* – uzayında sürekliliğin bir ayrışımı verildi. Eğer (X, τ, I) uzayının her açık A alt kümesi için (A)*⊂ A ise bu taktirde bu uzaya SA* –uzay denir. SA*-uzayında δ_I – r –süreklilik (sırasıyla, pre-I-süreklilik, semi-I-süreklilik, * – perfect süreklilik) ile R – I – sürekliliğin (sırasıyla, R – I – süreklilik, t – I – süreklilik, kendi içinde *-yoğun süreklilik) birbirine eşdeğer olduğu gösterildi.

Anahtar Kelimeler: R-I-açık küme, δ – I – açık küme, δ – I – regüler küme, R – I – sürekliliğin ayrışımı, ideal topoloji.

1. INTRODUCTION

Recently, ACIKGOZ et al. (2004) introduced the notion of a "δ – I – open set" in an ideal topological space, investigated some of its properties and obtained a decomposition of a α – I – continuous function using this set. HATIR & NOIRI

(2002) introduced the notions of t – I – sets, α* – I – sets, B_I – sets and C_I – sets. YÜKSEL et al. (2005) introduced the notion of an R – I – open set and obtained some of its properties.

The purpose of this paper is to introduce a decomposition of continuity on F* – spaces and also to show that δ_I – r – continuity (resp. pre – I – continuity, semi – δ – I – continuity, * – perfect continuity) is equivalent to R – I – continuity (resp. R – I – continuity, t – I – continuity, * – dense – in – itself continuity) if the domain is an SA* – space.

2. PRELIMINARIES

Let (X,τ) be a topological space, and A ⊂ X. Throughout this paper Cl(A) and Int(A) denote the closure and the interior of A with respect to τ, respectively.

An ideal, I is defined as a nonempty collection of subsets of X satisfying the following two conditions: (1) If A∈*I* and B ⊂ A, then B∈*I*; (2) If A∈*I* and B∈*I*, then A∪B∈*I*. An ideal topological space is a topological space (X,τ) with an ideal *I* on X and is denoted by (X,τ,*I*). For a subset A of X, A* (*I*) = {x∈X │ U∩A∉*I* for each neighborhood U of x} is called the local function of A with respect to *I* and τ (KESKİN et al. 2004). We simply write A* instead of A*(*I*) when there is no chance for confusion. Note that X* is often a proper subset of X. The hypothesis that X = X* (HATIR & NOIRI 2005) is equivalent to the hypothesis that τ ∩ *I* = Ø(Levine, 1963). The ideal topological spaces which satisfy this hypothesis are called *Hayashi – Samuels space*. (ANKOVIĆ & HAMLETT 1990). For every ideal topological space (X, τ, *I*), there exists a topology τ*(*I*), finer than τ, generated by β (I,τ) = {U \ I: U∈τ and I∈*I*}, but in general β (I,τ) is not always a topology (JANKOVIĆ & HAMLETT 1990).

Additionally, Cl*(A) = A∪A* defines a Kuratowski closure operator for τ*(*I*).

First we shall recall some definitions that will be used in the sequel.

DEFINITION 1. A subset A of an ideal topological space (X, τ) is said to be *regular open* (DUGUNDJI 1966) (*semi – open* (KURATOWSKI 1966)) if A = Int(Cl(A)) (A⊂Cl(Int(A))).

DEFINITION 2. A subset A of an ideal topological space (X, τ, I) is said to be
a) *α – I – open* (HATIR & NOIRI 2002)if A ⊂ Int(Cl*(Int(A))),
b) *α*– I – set* (HATIR & NOIRI 2002) if A = Int(Cl*(Int(A))),
c) *pre – I – open* (DONTCHEV 1996) if A ⊂ Int(Cl*(A)),
d) *R – I – open* (YUKSEL et. al. 2005) if A = Int(Cl*(A)),
e) *t – I – set* (HATIR & NOIRI 2002) if Int(A) = Int(Cl*(A)),
f) *δ – I – open* (ACİKGOZ et. al. 2004) if Int(Cl*(A)) ⊂ Cl*(Int(A)),
g) *regular I – closed* (SAMUELS 1975) if A=(Int(A))*,
h) *I – open* (ABD EL – MONSEF et. al. 1992) if A ⊂ Int((A)*),
i) f_I *– set* (KESKİN et. al. 2004) if A⊂(Int(A))*,
j) *semi – I – open* (HATIR & NOIRI 2002) if A⊂Cl*(Int(A)),

k) *β – I – open* (HATIR & NOIRI 2002) if A⊂Cl(Int(Cl*(A))),
l) * – *perfect* (HAYASHI 1964) if A=A*,
m) * – *dense – in – itself* (HAYASHI 1964) if A⊂A*,
n) *I – locally closed* (DONTCHEV 1999) if A=U∩V, where U is open and V is * – perfect,
o) *B_I – set* (HATIR & NOIRI 2002) if A=U∩V, where U is open and V is t – I – set,
p) *C_I – set* (HATIR & NOIRI 2002) if A=U∩V, where U is open and V is α*– I – set.

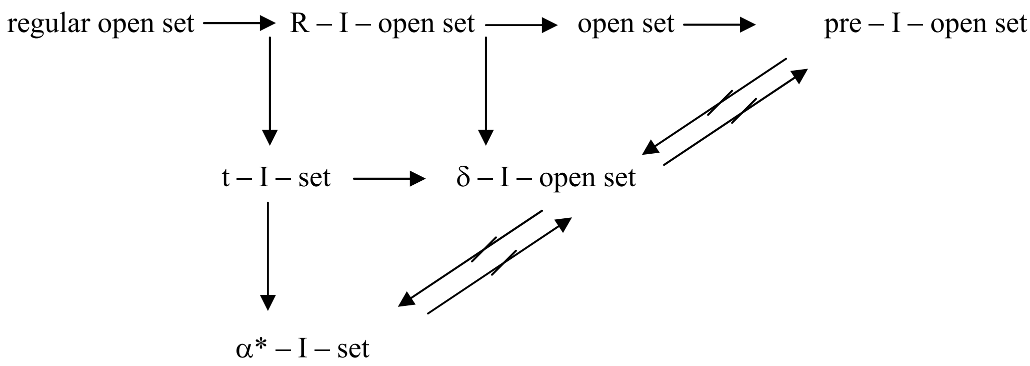
The family of all R – I – open (resp. α – I – open, pre – I – open, t – I – set, δ – I – open, * – perfect set, * – dense – in – itself) sets in an ideal topological space (X, τ, I) is denoted by RIO (X,τ) (resp. αIO (X,τ), PIO (X,τ), tIO (X,τ), δIO (X,τ), *PI (X,τ), *DI (X,τ)).

DEFINITION 3. A subset A of an ideal topological space (X, τ, I) is said to be *δ – I – regular* (ACIKGOZ &YUKSEL 2006) if A is both a pre – I – open set and a δ – I – open set.

The family of all δ – I – regular sets of (X, τ, I) is denoted by $\delta_I R$ (X,τ), when there is no chance for confusion with the ideal.

The following diagram is given by Acikgoz et al. (ACIKGOZ & YUKSEL 2006).

DIAGRAM I



3. ON F*– SPACES AND SA*– SPACES

PROPOSITION 1. Let (X,τ,I) be an ideal topological space and A a subset of X. Then the following properties hold:

a) If A is an R – I – open set and (X, τ, I) is a Hayashi-Samuels space, then A is an I – locally closed set,

b) If A is an R – I – open set, then A is a B_I – set.

c) If A is a B_I – set, then A is a C_I – set.

PROOF. a) Let A be an R – I – open set. Since (X,τ,I) is a Hayashi-Samuels space, then X* = X. Since every R – I – open set is an open set by (ACIKGOZ & YUKSEL 2006) and X is a * – perfect set, A = A∩X is an I – locally closed set.

b) Let A be an R – I – open set. Hence A is a t – I – set by (ACIKGOZ & YUKSEL 2006). Since X is an open set, $A = A \cap X$ is a B_I – set.

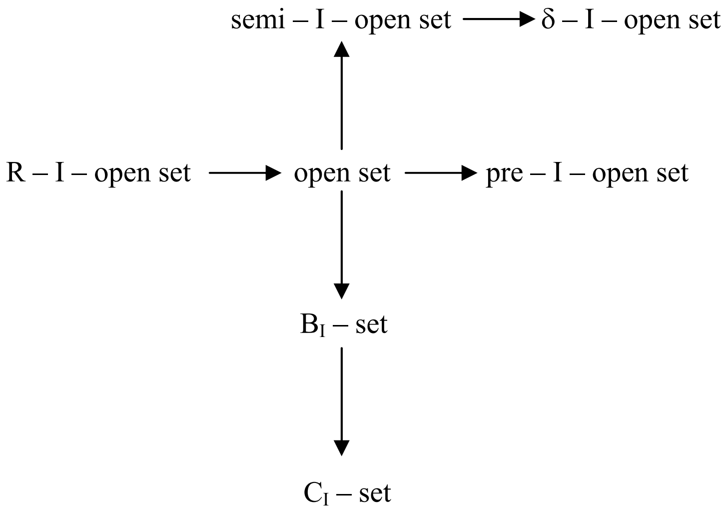
c) The proof is obvious from (HATIR & NOIRI 2002).

REMARK 1. The converse of Proposition 1(b) need not be true as shown in the following example. Also, HATIR & NOIRI (2002) showed that C_I – set is not a B_I – set, in general.

EXAMPLE 1. Let $X = \{a,b,c,d\}$, $\tau = \{\emptyset, X, \{a,b\}\}$, $I = \{\emptyset, \{c\}\}$. Set $A = \{d\}$. Then A is a B_I – set which is not an R – I – open set. For $A = \{d\}$, since $Cl^*(A) = \{c,d\}$, $Int(Cl^*(A)) = \emptyset$ and $Int(A) = Int(Cl^*(A))$, A is a t – I – set. By [7, Proposition 3.1(c)] since every t – I – set is a B_I – set, A is a B_I – set. On the other hand, since $Cl^*(Int(A)) = \emptyset \neq A$, A is not an R – I – open set.

DIAGRAM II has been expanded by using DIAGRAM I. ACİKGOZ et al. (2004) also have showed that every semi – I – open set is a δ – I – open set and definitions of pre – I – open sets and δ – I – open sets are independent concepts. HATİR & NOIRI (2002) have already showed that every open set is a B_I – set and every B_I – set is a C_I – set.

DIAGRAM II



PROPOSITION 2. For a subset, A of an ideal topological space, (X,τ,I), the following properties hold:
a) Every regular I – closed set is a δ – I – open set,
b) Every δ – I – regular set is a β – I – open set,
c) Every * – perfect set is a δ – I – open set.

PROOF. a) Let A be a regular I – closed set. Then $A = (Int(A))^*$ and so $A \subset Cl^*(Int(A))$. By Definition 2, A is a semi – I – open set. Hence, A is a δ – I – open set using Diagram II.

b) Let A be a δ – I – regular set. By Definition 3, A is a pre – I – open set. Then A ⊂ Int(Cl*(A)) ⊂ Cl(Int(Cl*(A))). Hence A is a β – I – open set.

c) Let A be a * – perfect set. Then A = (A)* and so Cl*(A) = A∪A* = A. Hence Int(Cl*(A)) = Int(A), and therefore, A is a t – I – set. Since every t – I – set is a δ – I – open set by Diagram I, A is δ – I – open set.

REMARK 2. The converses of Proposition 2 need not be true as shown in the following examples.

EXAMPLE 2. Let X = {a,b,c,d}, τ = {∅,X,{d},{a,c},{a,c,d}}, I = {∅,{a}}. Set A = {a,c}. Then A is a δ – I – open set which is not a regular I – closed set. For A = {a,c}, since Cl*(A) = {a,b,c}, Int(Cl*(A)) = {a,c}, Cl*(Int(A)) = Cl*({a,c}) = {a,b,c} and Int(Cl*(A)) ⊂ Cl*(Int(A)), A is a δ – I – open set. On the other hand, since A ⊄ Int(Cl*(A)), A is not regular I – closed set.

EXAMPLE 3. Let X = {a,b,c,d}, τ = {∅,X,{a},{c},{a,c}}, I = {∅,{a},{c}, {a,c}}. Set A = {c,d}. Then A is a β – I – open set which is not a δ – I – regular set. For A = {c,d}, since Cl*(A) = {b,c,d} Int(Cl*(A)) = {c}, Cl(Int(Cl*(A))) = {b,c,d} and so A ⊂ Cl(Int(Cl*(A))). This shows that A is not a β – I – open set.

EXAMPLE 4. Let X = {a,b,c,d}, τ = {∅,X,{c},{a,b,d}}, I = {∅,{d}}. Set A = {c,d}. Then A is a δ – I – open set which is not a * – perfect set. For A = {c,d}, since Cl*(A) = {c,d}, Int(Cl*(A)) = {c}, Cl*(Int(A)) = {c} and so Int(Cl*(A)) ⊂ Cl*(Int(A)). This shows that A is a δ – I – open set. On the other hand, since (A)* = {c} ≠ A, A is not a * – perfect set.

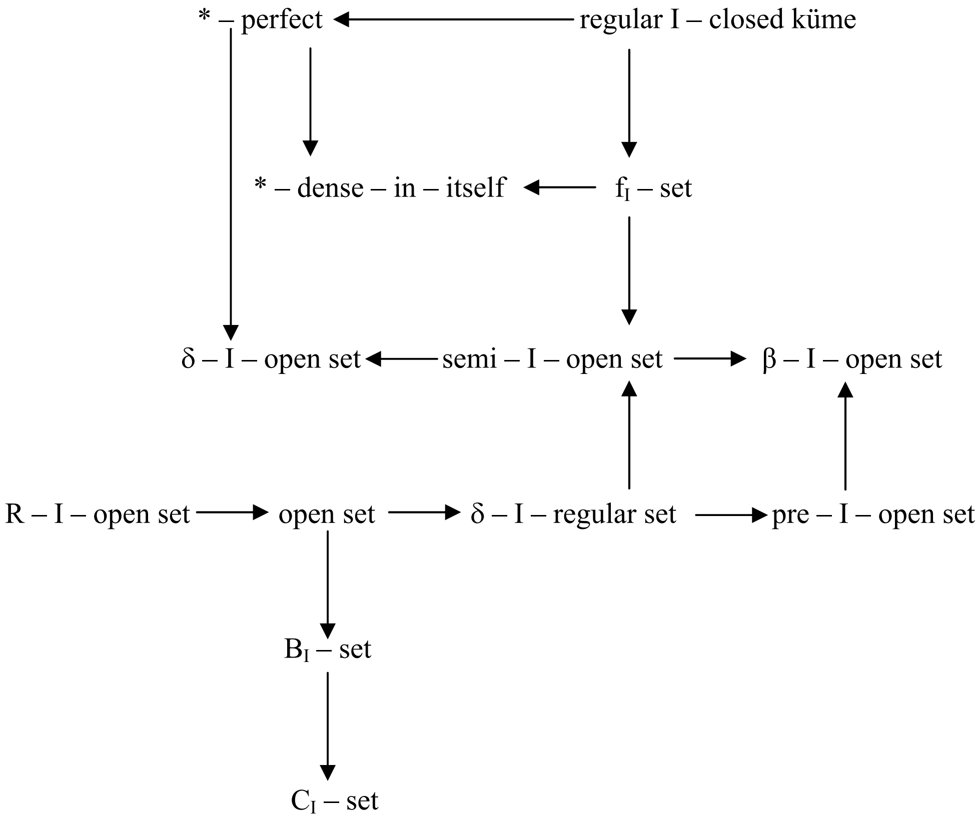
REMARK 3. Using the two examples presented below, it is shown that R – I – open sets and regular I – closed sets are independent of each other.

EXAMPLE 5. Let X = {a,b,c,d}, τ = {∅,X,{b},{a,c},{a,b,c}},I = {∅,{a},{d}, {a,d}}. Set A = {b}. Then A is a R – I – open set which is not a regular I – closed set. For A = {b}, since Cl*(A) = {b,d} and Int(Cl*(A)) = {b}. This shows that A is not an R – I – open set. On the other hand, since (Int(A))* = {b,d} ≠ A, A is not a regular I – closed set.

EXAMPLE 6. Let X = {a,b,c,d}, τ = {∅,X,{a},{b,c},{a,b,c}}, I = {∅,{a},{b}, {a,b}}. Set A = {b,c,d}. Then A is a regular I – closed set which is not an R – I – open set. For A = {b,c,d}, since (Int(A))* = {b,c,d}. This shows that A is a regular I – closed set. On the other hand, since Cl*(Int(A)) = {b,c,d} ≠ A, A is not an R – I – open set.

REMARK 4. The following diagram showing the relationship among several sets defined above, is obtained using DIAGRAM II, Proposition 2 and the Diagram in (KESKİN et. al. 2004).

DIAGRAM III



DEFINITION 4. An ideal topological space (X,τ,I) is said to be a *F* – space* if A=Cl*(A) for every open set A ⊂ X.

EXAMPLE 7. Let X = {a,b,c}, τ = {∅,X,{c}}, I = {∅,{a},{c},{a,c}}. Then X is a F* – space. For every A∈τ, we have since (A)* ⊂ A.

EXAMPLE 8. Let X = {a,b,c,d}, τ = {∅,X,{c},{a,c},{b,c},{a,b,c},{a,c,d}}, I = {∅,{a},{b},{a,b}}. For every A∈τ, since (A)* ⊄ A, we have X is not F* – space.

DEFINITION 5. A subset A of an ideal topological space (X,τ,I) is said to be
a) A semi – I – closed if (HATIR & NOIRI 2005) Int(Cl*(A)) ⊂ A,
b) A *SC – I – open set* if A = U ∩ V, where U∈τ and A is semi – I – closed set.

THEOREM 1. For a subset A of an ideal topological space (X, τ, I), the following property holds: A is an open set if and only if A is an α – I – open set and a SC – I – open set.

PROOF. The necessity is obvious.
Sufficiency: Let A be an α – I – open and a SC – I – open set. Then A ⊂ Int(Cl*(Int(A))) and A = U ∩ V, where U ∈ τ and V is semi – I – closed.

We have $A \subset Int(Cl^*(Int(A))) = Int(Cl^*(Int(U \cap V))) \subset Int(Cl^*(U \cap V)) \subset Int(Cl^*(U)) \cap Int(Cl^*(V)) = Int(Cl^*(U)) \cap Int(V)$. Thus we obtain $A = U \cap A \subset U \cap Int(Cl^*(U)) \cap Int(V) = Int(U \cap V) = Int(A)$ and hence A is open – set.

PROPOSITION 3. For a subset, A of a F* – space (X,τ,I), the following properties hold: A is an open set if and only if A is a pre – I – open set and a δ – I – open set.

PROOF. Necessity: The proof is obvious from DIAGRAM I.
Sufficiency: Let A be a pre – I – open set. Then $A \subset Int(Cl^*(A))$. Since A is a δ – I – open set, $Int(Cl^*(A)) \subset Cl^*(Int(A))$. Furthermore by hypothesis, since X is also an F* – space, $Cl^*(Int(A)) \subset Int(A)$. Consequently, $A \subset Int(Cl^*(A)) \subset Cl^*(Int(A)) \subset Int(A)$, that is, A = Int(A) and hence A is an open set.

DEFINITION 6. An ideal topological (X,τ,I) is said to be an *SA* – space* if $(A)^* \subset A$ for every set $A \subset X$.

EXAMPLE 9. Let X = {a,b,c}, τ = {∅,X,{a,c}}, I = {∅,{a},{c},{a,c}}. Then X is an SA* – space. For every $A \subset X$ since $(A)^* \subset A$, X is an SA* – space.

THEOREM 2. Every SA* – space is F* – space.

REMARK 5. The converse of Theorem 2 need to be true as Example 7 shows.

In any ideal topological space (X,τ,I) SA* – space, we have the following fundamental relationships between the classes of subsets of X considered:

PROPOSITION 4. For a subset A of an ideal topological space (X,τ,I) SA* – space, the following properties hold:
a) $\delta_I R$ (X,τ) = RIO (X,τ),
b) tIO (X,τ) = δIO (X,τ),
c) *PI (X,τ) = *DI (X,τ).

PROOF. a) Necessity: By [3, Proposition 3(a)], we have RIO (X,τ) $\subset \delta_I R$ (X,τ).
Sufficiency: Let A be a δ – I – regular set. According to Definition 3, A is both a δ – I – open set and a pre – I – open set. Hence $A \subset Int(Cl^*(A)) \subset Cl^*(Int(A))$. Furthermore by hypothesis, since X is also an SA* – space, $(Int(A))^* \subset Int(A)$, $Cl^*(Int(A)) = (Int(A) \cup (Int(A)^*)) \subset Int(A)$ and $Cl^*(Int(A)) \subset Int(A)$. Consequently, $A = Int(Cl^*(A))$ and hence A is an R – I – open set.

b) and c) are analogous to the Proof of (a) and are thus omitted.

PROPOSITION 5. For a subset A of an SA* ideal topological space (X,τ,I), the following properties hold:
a) Every I – open set is an R – I – open set,
b) Every f_I – set is an open set,
c) Every β – I – open set is a semi – open set.

PROOF. a) Let A be an I – open set. Then $A \subset Int((A)^*)$. Furthermore by hypothesis, since X is also an SA* – space, $(A)^* \subset A$ and $Cl^*(A) \subset A$. Consequently, $A \subset Int((A)^*) \subset Int(Cl^*(A)) \subset Int(A) \subset A$ and so $A = Int(Cl^*(A))$. Hence A is an R – I – open set.
b) Let A be an f_I – set. By Definition 2, we have $A \subset (Int(A))^*$. Since X is also an SA* – space, $(Int(A))^* \subset Int(A)$ and so $A \subset (Int(A))^* \subset Int(A)$. Hence A is an open set.
c) Let A be an β – I – open set. Then $A \subset Cl(Int(Cl^*(A)))$. Since X is an SA* – space, $A \subset Cl(Int(Cl^*(A))) \subset Cl(Int(A))$ and so $A \subset Cl(Int(A))$. Hence A is a semi – open set.

DEFINITION 7. A function $f : (X, \tau) \to (Y, \varphi)$ between two topological spaces is said to be *semi – continuous* (KURATOWSKI 1966) if for every $V \in \varphi$, $f^{-1}(A)$ is semi – open of (X, τ).

DEFINITION 8. A function $f : (X, \tau, I) \to (Y, \varphi)$ from an ideal topological space to a topological space is said to be *R – I – continuous* (ACIKGOZ & YUKSEL 2006) (resp. *δ_I – r – continuous* (ACIKGOZ & YUKSEL 2006), *pre – I – continuous* (DONTCHEV 1996), *semi – δ – I – continuous* (ACİKGOZ et. al. 2004), *I – continuous* (ABD EL – MONSEF et. al. 1992), *f_I – I – continuous* (KESKIN et. al. 2004), *α*– I – continuous* (HATIR & NOIRI 2002), *t – I – continuous* (HATIR & NOIRI 2002), ** – perfect continuous* (HAYASHI 1964),** – dense – in – itself continuous* (HAYASHI 1964)) if for every $V \in \varphi$, $f^{-1}(A)$ is R – I – open (resp. δ – I – regular, pre – I – open, α*– I – set, t – I – set) of (X, τ, I).

We are now able to provide a decomposition of continuity in this setting.

THEOREM 3. Let (X, τ, I) be an F* – space. For a function $f : \to (X, \tau, I)$ (Y, φ), the following properties are equivalent:
a) f is continuous,
b) f is pre – I – continuous and semi – δ – I – continuous.

PROOF. This follows from Proposition 3.

THEOREM 4. Let (X, τ, I) be an SA* – space. For a function $f : (X, \tau, I) \to (Y, \varphi)$, the following equivalences hold:
a) f is δ_I – r – continuous if and only if it is R – I – continuous,
b) f is semi – δ – I – continuous if and only if it is t – I – continuous,
c) f is * – perfect continuous and if and only if it is * – dense – in – itself continuous.

PROOF. This follows from Proposition 4.

THEOREM 5. Let (X, τ, I) be an SA* – space. For a function $f : (X, \tau, I) \to (Y, \varphi)$, the following implications hold:
a) If f is I – continuous, then it is R – I – continuous,
b) If f is f_I – continuous, then it is continuous,
c) If f is β – I – continuous, then it is semi – continuous.

PROOF. This follows from Proposition 5.

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